# Column Basis Reduction and Decomposible Knapsack Problems

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# Outline of the Presentation

① Basis reduction

② Column Basis Reduction (CBR)

- CBR in Range Space
- CBR in Null Space
- CBR with rhs reduction

③ Branching on a constraint

④ Decomposible knapsack problems

# What is Column Basis Reduction?

Given integral matrix A, compute unimodular U s.t.

- ➜ columns of AU have small Euclidean norm
- ➜ and nearly orthogonal (angle between any column and the linear space spanned by other columns is $\geq 60$ degrees)

Methods are:

- LLL-reduction by Lenstra, Lenstra and Lovasz
- Korkhine-Zolotarev (KZ) reduction

A= $\begin{pmatrix} 289 & 18 \\ 466 & 29 \\ 273 & 17 \end{pmatrix}, U = \begin{pmatrix} 1 & -15 \\ -16 & 241 \end{pmatrix}, AU = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}$

## THE OUTLINE OF LLL-BR METHOD

Given matrix $B \in \mathbb{R}^{mxn}$ with m independent columns, $\mathbb{L}(B) = \{Bv|v \in \mathbb{Z}^n\}$ Basically, set of all integral combinations of the columns of B.

- Finding shortest vector in L is believed to be NP-complete.
- LLL latice basis reduction is approximation algorithm in polynomial time

The Algorithm:

1. Find Gram-Schmidt basis of columns of $B$. Let them be $b_1^*, .., b_m^*$.

$$b_1^* = b_1 \, , b_k^* = b_k - \sum_{j=1}^{k-1} \mu_{kj} b_j^* , \;\; \mu_{kj} = \frac{b_k . b_j^*}{b_j^* . b_j^*}$$

## The Algorithm: Cont.

2. $b_1, b_2, ..b_m$ is reduced if $|\mu_{kj}| \leq 1/2$ for $1 \leq j < k \leq m$ and

$$b_k^*.b_k^* \geq (\alpha - \mu_{kk-1}^2)b_{k-1}^*.b_{k-1}^* \quad (**)$$

for $1 < k \leq m$ and $1/4 < \alpha \leq 1$
We say that $b_k$ is size-reduced if $|\mu_{kj}| \leq 1/2$ for $1 < j \leq k$

3 Let $b_k \leftarrow b_k - \lceil \mu_{kk-1} \rfloor b_{k-1}$.
If (**) holds, size-reduce $b_k$ completely, do
$b_k \leftarrow b_k - \lceil \mu_{kj} \rfloor b_j$ for $j = k-2, ..1$, increment k

4 Otherwise swap $b_k$ and $b_{k-1}$, decrement k.

# Column Basis Reduction (CBR)

## CBR in the Range Space:

- Change

$$\begin{aligned} (IP) b' &\leq Ax \leq b \\ x &\in \mathbb{Z}^n \end{aligned}$$

- to

$$\begin{aligned} (\widetilde{IP}) b' &\leq AUy \leq b \\ y &\in \mathbb{Z} \end{aligned}$$

- The relation between x and y is $Uy = x$, $y = U^{-1}x$

Example: The infeasible problem,

$$
\begin{aligned}
106 \leq 21x_1 + 19x_2 \leq 113 \\
0 \leq x_1, x_2 \leq 6 \\
x_1, x_2 \in \mathbb{Z}
\end{aligned}
$$

- Branching on either variable will create at least 5 feasible nodes.

Apply CBR:

$$
\begin{aligned}
106 \leq -2y_1 + 7y_2 \leq 113 \\
0 \leq -y_1 - 6y_2 \leq 6 \\
0 \leq y_1 + 7y_2 \leq 6 \\
y_1, y_2 \in \mathbb{Z}
\end{aligned}
$$

- Branching on either variable y1 would create 4 feasible branches, but brancing on y2 immediately proves infeasibility.

CBR in the Null Space: Let $A_1 x = b_1$ be a system of equalities in $b' \leq Ax \leq b$.

- compute an integral matrix $B_1$ and and integral vector $x_0$ such that $\{x \in \mathbb{Z}^n | A_1 x = b_1\} = \{B_1 \lambda + x_0 | \lambda \in \mathbb{Z}^{n-m_1}\}$
- $A_1 x_0 = b_1$ and $A_1 B_1 = 0$.
- $B_1$ and $x_0$ computed with Hermite Normal Form computation.
- Substitute $B_1 \lambda + x_0$ for $x$ in original problem and apply CBR in range space.

## CBR in the Null Space:

- Change

$$
\begin{aligned}
Ax &= b \\
l \le x &\le u \\
x &\in \mathbb{Z}^n
\end{aligned}
$$

- to

$$
\begin{aligned}
l \le B_1\lambda + x_0 &\le u \\
\lambda &\in \mathbb{Z}^{n-m_1}
\end{aligned}
$$

- And apply CBR in Range Space.

## CBR with Right Hand Side Reduction:

- On several instances RHS reduction gives better reformulations. Write IP as

$$\begin{aligned} Dx &\leq f \\ x &\in \mathbb{Z}^n \end{aligned}$$

- Reformulate as

$$\begin{aligned} (DU)y &\leq f - Dx_r \\ y &\in \mathbb{Z}^n \end{aligned}$$

- $x_r$ is calculated (with Babai's algorithm) s. t. $Dx_r$ is an approximation to a closest vector to $f$ in the $\mathbb{L}(D)$.

# Branching on a Constraint

Given P and integral vector c, the width of P in the direction of c is

- ➜ $width(c,P) = max\{cx|x \in P\} - min\{cx|x \in P\}$
- ➜ Branching on $cx$ means creating $cx = \lceil min \rceil, cx = \lceil min \rceil + 1, .., cx = \lfloor max \rfloor$ branches
- ➜ If $[min, max]$ does not contain any integer then $P$ is infeasible
- ➜ If $c$ is unit vector then it is regular $x$ branching.

## Example

$$106 \le 21x_1 + 19x_2 \le 113$$
$$0 \le x_1, x_2 \le 6$$
$$x_1, x_2 \in \mathbb{Z}$$

- Branching on $x1 + x2$ will immediately prove the problem is infeasible, since $min = 5.04$ and $max = 5.94$.

# T+1-LEVEL DECOMPOSIBLE KNAPSACK PROBLEM

Assume

1. Given matrix $P \in \mathbb{Z}^{txn}$, row vectors $a, r \in \mathbb{Z}^n$, a column vector $u \in \mathbb{Z}^n_{++}$. $u$ might have components equal to $+\infty$. and $p_i$ represent a row of $P$.
2. Given a row vector $M \in \mathbb{Z}^t_{++}$ with $M_1 > M_2 > .. > M_t$
3. $a = MP + r$

*Definition:* The feasibility problem

$$\begin{aligned} \beta' &\leq ax \leq \beta \\ 0 &\leq x \leq u \\ x &\in \mathbb{Z}^n \end{aligned}$$

is called t+1-level decomposible knapsack problem.

# 2-LEVEL DECOMPOSIBLE KNAPSACK PROBLEM

$$\begin{aligned} \beta' &\leq ax \leq \beta \\ 0 &\leq x \leq u \\ x &\in \mathbb{Z}^n \end{aligned}$$

where

- ➜ $a = pM + r$ with $p \in \mathbb{Z}^n_+$, $r \in \mathbb{Z}^n$; M large
- ➜ $\beta, \beta'$ are chosen, so the instance is LP-feasible.
- ➜ IP-infeasibility can be proven by branching $px$
- ➜ The previous example is 2-level decomposible knapsack problem with $p = (1, 1)$, $r = (1, -1)$, $u = (6, 6)$, $M = 20$, $a = pM + r = (21, 19)$
- ➜ Remember, branching on $px = x_1 + x_2$ proves infeasibility at root node

# Reformulation with CBR in Range Space

Calculate $U$ such that

A= $\begin{pmatrix} a \\ I \end{pmatrix} = \begin{pmatrix} pM + r \\ I \end{pmatrix}$ U is reduced.

**Theorem1:** If M is sufficiently large then

$pU = (0, 0, .., \alpha)$ for some $\alpha \in \mathbb{Z} \backslash \{0\}$

**Corrollary:** $Uy = x \Rightarrow pUy = px \Rightarrow \alpha y_n = px$

$\Rightarrow$ branching on $y_n$ proves infeasibility.

Sufficiently large means

- If LLL (Lenstra,Lenstra,Lovasz) reduction is used, $M > 2^{n+1} \| p \| \| r \|^2$.
- If KZ (Korkhine-Zolotarev) reduction is used, $M > \sqrt{n} \| p \| \| r \|^2$.

Strength of the BR algorithm is represented by $c_n$.
$c_n(LLL) = 2^{n+1}$ and $c_n(KZ) = \sqrt{n}$.
If $c_n$ is smaller, the columns are more reduced.

Let A= $\begin{pmatrix} a \\ I \end{pmatrix} = \begin{pmatrix} MP + r \\ I \end{pmatrix}$, $\widetilde{P} = PU$ and $\widetilde{p_i}$ for the rows of $\widetilde{P}$.

**Theorem2:** *There exists functions $f_1, f_2, ..f_t$ with:*

*(1) Given $s \in \mathbb{Z}^t$ with $1 \leq s_t \leq ... \leq s_1 \leq n - t$*
*If*

$$M_i > f_i(M_{i+1}, .., M_t, s_i, P, r, c_n) \ (i = 1, .., t) \text{ (*)}$$

*then*

$$\widetilde{p}_{i,1:s_i} = 0 \ (i = 1, .., t)$$

*(2) There is $M$ with*

$$size(M) = poly(size(P), size(r), size(c_n), n)$$

*that satisfies (*).*

## What Theorem 2 says:

- If $M_1$ is sufficiently large compared to $M_2, ..M_t$, then $p_1 M_1$ contributes the most to the length of $a$.
- If $M_2$ is sufficiently large compared to $M_3, ..M_t$, then $p_2 M_2$ contributes the second most to the length of $a$ and so on.
- To reduce the length of the columns of $A$, zero out many components of $p_1$, fewer components of $p_2$ and so on.
- Let $n = 10$, $t = 4$, $s_1 = 6$, $s_2 = s_3 = 5$, $s_4 = 4$, the matrix $\widetilde{P}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * & * \end{pmatrix}$$

## BRANCHING IN CBR AND B &B

Since $\widetilde{p}_i y = p_i x \ (i = 1, .., t)$

- ➜ Branching on $y_n, ..y_{s_1+1}$ in CBR in Range space has the same effect as branching on $p_1 x$ in original problem.
- ➜ Branching on $y_{s_1+1}, .., y_{s_2}$ in CBR in Range space has the same effect as branching on $p_2 x$ in original problem.

Thus, CBR

- ➜ takes the unkown dominant branching combinations
- ➜ transforms them into individual variables s.t. $y_n$ has more significance than $y_{n-1}$

Example:

- An instance of 3-level knapsack problem with $n = 11$,$t = 2$, $u = e$, $M_1 = 220$, $M_2 = 11$, $\beta = 5661$, $\beta' = 5660$
- $p_1 = (2, 3, 5, 7, 8, 8, 9, 10, 10, 11, 11)$,
  $p_2 = (7, 6, 5, 3, 3, 6, 4, 2, 6, 4, 7)$,
  $r = (3, -1, 1, 1, 1, 3, -1, -1, 1, 1, -1)$
- $$5660 \leq 520x_1 + 725x_2 + 1156x_3 + 1574x_4 + 1794x_5 + 1829x_6 + 2023x_7 + 2221x_8 + 2267x_9 + 2465x_{10} + 2496x_{11} \leq 5661$$
  $$x_i \in \{0, 1\}$$

## Example: Cont.

- It is reasonably hard with pure B&B on $x_i$ variables. Few hundred nodes to prove infeasibility
- Branching on $p_1 x$ is very useful. $24.30 \leq p_1 x \leq 25.34$
- Then, branching on $p_2 x$ proves infeasibility, since $14.02 \leq p_2 x \leq 14.93$
- After CBR, branch on $y_{11} = 25$ that results $24.30 \leq y_2 \leq 25.34$, then branching on $y_{10}$ proves infeasibility since $14.02 \leq y_{10} \leq 14.93$

# COMPUTATIONAL RESULTS

If maximization (or minimization) problem:

$$
\begin{aligned}
max \;\; cx \qquad & \\
b' \;\; \leq & \;\; Ax \leq b \\
x \;\; \in & \;\; \mathbb{Z}^n
\end{aligned}
$$

Apply CBR to $\begin{pmatrix} c \\ A \end{pmatrix}$

The CBR is successful in the following problems:

- Subset sum problems
- Strongly correlated knapsack problems
- The market share problems