Using Split Disjunctions to Improve the Formulation of MILP

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References

- K. Andersen, Using split disjunctions to improve the formulation of a mixed integer linear program.

Introduction

- $\min\{c^T x : Ax \geq b, x \geq 0$ and x_j integer for $j \in N_I\}$.
- $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$ and $N_I \subseteq N := \{1, \ldots, n\}$.
- The set P denotes the feasible solution to LP.
- The set P_I denotes the feasible solution to MILP.

Introduction

- Problem formulation (matrix A and vector b) is crucial for solving MILP.
- Solvers cannot expect users to provide a “good” formulation.
- Why not try to replace (A, b) with better formulation (A', b')?
- The idea is try to replace a constraint with a valid inequality that strictly dominates it.
- The advantage is number of constraints is not changed.

Good Formulation

- $\alpha^T x \geq \beta$ dominates $a_i^T x \geq b_i$, if replacing $a_i^T x \geq b_i$ with $\alpha^T x \geq \beta$ gives tighter LP relaxation, $R \subseteq P$.
- Replace $a_i^T x \geq b_i$, where $i \in M$ with a valid inequality for P_I that dominates it on P.
- $\alpha^T x \geq \beta$ strictly dominates if $\alpha^T y' < \beta$ for some $y' \in P$.
- Consider relaxations of P_I derived from split disjunctions.

Disjunctive Constraints

- MILP can be formulated as LP in which the feasible solution have to satisfy a set of disjunction (Balas).
- A set $S = S^1 \cup S^2$ with $S^i \subseteq \mathbb{R}^n_+$ is a disjunction (union) of the two sets S^1 and S^2.
- If $\sum_{j=1}^n \pi_j^1 x_j \leq \pi_0^1$ is valid for S^1 and $\sum_{j=1}^n \pi_j^2 x_j \leq \pi_0^2$ is valid for S^2, then $\sum_{j=1}^n \min(\pi_j^1, \pi_j^2) x_j \leq \max(\pi_0^1, \pi_0^2)$ is valid for S.

Disjunctive Procedure

- Disjunctive procedure is a systematic way to generate valid inequalities for $S = x \in \mathbb{Z}^n_+ : Ax \leq b$.
- Given $\delta \in Z^1_+$, if

◇ $\sum_{j=1}^{n} \pi_j x_j - \alpha(x_k - \delta) \leq \pi_0$ is valid for S for some $\alpha \geq 0$ and

◇ $\sum_{j=1}^{n} \pi_j x_j + \beta(x_k - \delta - 1) \leq \pi_0$ is valid for S for some $\beta \geq 0$ then

◇ $\sum_{j=1}^{n} \pi_j x_j \leq \pi_0$ is valid for S.

Example

- $P = \{x \in \mathbb{R}^2_+ : -x_1 + x_2 \leq \frac{1}{2}, \frac{1}{2}x_1 + x_2 \leq \frac{5}{4}, x_1 \leq 2\}$.
- The first two inequalities can be written as

◇ $-\frac{1}{4}x_1 + x_2 - \frac{3}{4}x_1 \leq \frac{1}{2}$

◇ $-\frac{1}{4}x_1 + x_2 + \frac{3}{4}(x_1 - 1) \leq \frac{1}{2}$

- Using the disjunction $x_1 \leq 0$ or $x_1 \geq 1$.
- The valid inequality is $-\frac{1}{4}x_1 + x_2 \leq \frac{1}{2}$ for $S = P \cup Z^2$.

Split Disjunction $D(\pi, \pi_0)$

- $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\pi_j = 0$ for $\forall j \notin N_I$.
- $D(\pi, \pi_0)$ splits the polyhedron in two parts (Picture),
  - ◇ $P_1^{(\pi,\pi_0)} = \{x \in P : \pi^T x \leq \pi_0\}$
  - ◇ $P_2^{(\pi,\pi_0)} = \{x \in P : \pi^T x \geq \pi_0 + 1\}$

Split Disjunction $D(\pi, \pi_0)$

- Any feasible solutions to MILP have to satisfy every split disjunction.
- Split disjunctions are used to generate cutting planes that cut off points of P violating $D(\pi, \pi_0)$.

Split Cut and Split Closure

- A split cut is a valid inequality for $Conv(P_1^{(\pi,\pi_0)} \cup P_2^{(\pi,\pi_0)})$.
- A split closure is the set of points in P that satisfy all split cuts.
- The split closure is a polyhedron (Cook, Kannan and Shrijver).

Recall

- $\min\{c^T x : Ax \geq b, x \geq 0 \text{ and } x_j \text{ integer for } j \in N_I\}$ (MILP)
- $\min\{c^T x : Ax \geq b, x \geq 0\}$ (LP)

◇ $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$,

◇ $M := \{1, \ldots, m\}$ and $N_I \subseteq N := \{1, \ldots, n\}$.

◇ The set P_I denotes the feasible solution to MILP.

◇ The set P denotes the feasible solution to LP.

- $D(\pi, \pi_0)$ is a splits disjunction.

◇ $P_1^{(\pi,\pi_0)} = \{x \in P : \pi^T x \leq \pi_0\}$

◇ $P_2^{(\pi,\pi_0)} = \{x \in P : \pi^T x \geq \pi_0 + 1\}$

Sequential Strengthening

⋆ Sequential: Improve the coefficient of one variable in a given constraint.

- Let

◇ $a_i^T x \geq b_i$ be an arbitrary constraint, where $i \in M$;

◇ x_k be an arbitrary variable, where $k \in N$;

- Aim: Find minimum δ such that $\sum_{j \in N \setminus \{k\}} a_{i,j} x_j + \delta x_k \geq b_i$ is valid for $R^{(\pi,\pi_0)} := Conv(P_1^{(\pi,\pi_0)} \cup P_2^{(\pi,\pi_0)})$.

Example

- $P = \{x \in \mathbb{R}^2_+ : x_1 - x_2 \geq -\frac{1}{2}, -\frac{1}{2}x_1 - x_2 \geq \frac{5}{4}, -x_1 \geq -2\}$.
- Using the splits disjunction $D(1,0,0)$ that is $x_1 \leq 0$ or $x_1 \geq 1$.
- (Picture)
- Find minimum δ such that $\delta x_1 - x_2 \geq -\frac{1}{2}$ is valid for $R^{(1,0,0)}$.

How to find δ?

- Find minimum δ^1 such that $\sum_{j\in N\setminus\{k\}} a_{i,j}x_j + \delta x_k \geq b_i$ is valid for $P_1^{(\pi,\pi_0)}$.
- Find minimum δ^2 such that $\sum_{j\in N\setminus\{k\}} a_{i,j}x_j + \delta x_k \geq b_i$ is valid for $P_2^{(\pi,\pi_0)}$.
- If one or both of δ^1 and δ^2 are found, then by disjunctive argument of Balas (*Disjunctive Programming. Annals of Discrete Mathematics, 5:3-51, 1979*), $\delta = \max(\delta^1, \delta^2)$

How to find δ^1?

- Solve $(LP_1(i,k))$

$$\begin{array}{rc} \max & \lambda b_i - \sum_{j \in N \setminus \{k\}} a_{i,j} x_j \\ \text{s.t.} & Ax \geq \lambda b \\ & x_k = 1, \\ & \lambda \geq 0, \\ & x \geq 0, \\ & \pi^T x \leq \lambda \pi_0. \end{array}$$

- Find the optimal objective value δ^1 valid for $P_1^{(\pi,\pi_0)}$

How to find δ^2?

- Solve $(LP_2(i,k))$

$$
\begin{array}{ll}
\max & \lambda b_i - \sum_{j \in N \setminus \{k\}} a_{i,j} x_j \\
\text{s.t.} & Ax \geq \lambda b \\
 & x_k = 1, \\
 & \lambda \geq 0, \\
 & x \geq 0, \\
 & \pi^T x \geq \lambda(\pi_0 + 1).
\end{array}
$$

- Find the optimal objective value δ^2 valid for $P_2^{(\pi,\pi_0)}$

Sequential Strengthening (Cont.)

- If both $(LP_1(i,k))$ and $(LP_2(i,k))$ are infeasible, then $x_k = 0$ for all $x \in R^{(\pi,\pi_0)}$.
- Otherwise the inequality $\sum_{j \in N \setminus \{k\}} a_{i,j} x_j + \hat{\delta} x_k \geq b_i$ is valid for $R^{(\pi,\pi_0)}$, where $\hat{\delta} = \max(\delta^1, \delta^2) \leq a_{i,k}$

Do we need to solve $LP_1(i,k)$ and $LP_2(i,k)$ every time?

- Let $LP'(i,k)$ denote the linear program obtained from $LP_1(i,k)$ ($LP_2(i,k)$) by deleting the disjunction constraint.
- *Lemma*: Let $\hat{x}$ be an optimal basic feasible solution to LP, and let (x^b, λ^b) be an arbitrary basic feasible solution to $LP_1(i,k)$
  1. If $\hat{x}_k > 0$ then $(\hat{y}, \hat{\lambda}) := (\frac{\hat{x}}{\hat{x_k}}, \frac{1}{\hat{x_k}})$ is a basic feasible solution to $LP'(i,k)$.
  2. If $\hat{x}_k > 0$ and $a_i^T \hat{x} = b_i$, then $(\hat{y}, \hat{\lambda})$ is an optimal solution to $LP'(i,k)$.
  3. If $a_q^T x^b = \lambda^b b_q$ where $q \in M$, then (x^b, λ^b) is an optimal solution to $LP_1(q,k)$.

Do we need to solve $LP_1(i,k)$ and $LP_2(i,k)$ every time?

- (1) and (2) indicate we can solve LP, find optimal basis to $LP'(i,k)$, then add disjunction constraint to re-optimize to get optimal solution to $LP_1(i,k)$ and $LP_2(i,k)$.
- (3) indicates any basic feasible solution to $LP_1(i,k)$ might be an optimal solution to $LP_1(q,k)$, where $q \in M$.

Experiment and Conclusion

- Compare two cutting plane (Mixed Interger Gomory cuts) algorithms, with and without strengthening.

◇ After 20 rounds of adding cutting-planes, call branch-and-bound algorithm to solve the problem.

◇ Use CPLEX 8.0. Test problem from MIPLIB 3.0

⋆ Strengthening helps in closing the gap on the MILP problems.

⋆ Number of branch-and-bound nodes are reduced.

⋆ Seems only improving the solution time on difficult problems.