Bicriteria Programming & Zero-sum Stackelberg Games

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Outline

Math Programming Generalization

Consider the mathematical programming problem

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{subject to} \quad & g(x,y) \leq 0 \end{aligned} \tag{1}$$

Now, suppose we would like constrain y to be an optimal solution to the mathematical program

$$\begin{aligned} \min_{y} \quad & f(x,y) \\ \text{subject to} \quad & h(x,y) \leq 0. \end{aligned} \tag{2}$$

Bilevel Programming Formulation

To model this situation, we can add the constraint

$$y \in \operatorname{argmin}\{f(x,y) : h(x,y) \leq 0\}$$

to (1). This yields the (continuous) bilevel programming problem (BPP):

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{subject to} \quad & g(x,y) \leq 0 \\ & y \in \operatorname{argmin}\{f(x,y) : h(x,y) \leq 0\} \end{aligned} \tag{3}$$

This is also a mathematical program, with a specially-structured nonlinear constraint. It is known to be $\mathcal{NP}$-hard, even if all functions are linear (Calamai and Vincente, 1994; Jeroslow, 1985; Ben-Ayed and Blair, 1990; Hansen et al., 1992).

Characteristics of Bilevel Programs

Bilevel programs can generally characterized by;

- Combination of two mathematical programs where one is contained in the constraint set of the other
- Hierarchical relationship, since one program must be evaluated be fore we can evaluate the other
- One decision maker has control over all variables

Stackelberg Game

A *Stackelberg Game* is defined by:

- Two (or more) players, where one of the players is a *leader* and the other a *follower*
- Leader moves first, follower reacts to leader's decision
- If the game is played once, we call it a *static* game. If we repeat a static game multiple times, we call it a *dynamic* game.

We usually assume:

- Perfect information - follower is aware of the leader's action
- Rationality - neither player will choose a suboptimal strategy

Static Stackelberg Problem

If the optimal strategies of the players in a static Stackelberg game are solutions to a mathematical program, we can model the game by:

$$\begin{aligned} \min_{x} \quad & F(x, y) \\ \text{subject to} \quad & g(x, y) \leq 0 \\ & y \in \operatorname{argmin} \{ f(x, y) : h(x, y) \leq 0 \} \end{aligned} \tag{4}$$

called the static Stackelberg Problem (SSP). SSP is related to BPP. Note that in this problem, the leader (DM) only has control over the x variables.

Comment

It is assumed that the leader has *perfect information* about how the follower chooses among alternative optima to the subproblem, if they exist.

Zero-sum Stackelberg Game

Suppose we have

$$F(x,y) = -f(x,y)$$

then the game is *zero-sum*. Applying this to SSP yields the *zero-sum static Stackelberg game* (ZSSP):

$$\begin{aligned} \min_{x} \quad & f(x,y) \\ \text{subject to} \quad & g(x,y) \leq 0 \\ & y \in \operatorname{argmax}\{f(x,y) : h(x,y) \leq 0\} \end{aligned} \tag{5}$$

Comment

If all functions in (5) are linear, this is called the linear maxmin problem (LMM).

A Natural Generalization

The most natural generalization of all problems described is to allow integrality constraints on some or all of the variables. This yields the mixed-integer zero-sum static Stackelberg problem (MZSSP):

$$\begin{aligned} \min_{x} \quad & f(x,y) \\ \text{subject to} \quad & g(x,y) \leq 0 \\ & x \in X_{INT} \\ & y \in \operatorname{argmax}\{f(x,y) : h(x,y) \leq 0, y \in Y_{INT}\} \end{aligned} \tag{6}$$

where X_{INT} and Y_{INT} represent integrality constraints on a subset of the leader and follower variables, respectively.

A Special Case

Let's consider the special case of (6) where:

- $X_{INT} = \{0, 1\}$
- The leader's constraint set contains the budget constraint $b(x, y) \leq B$
- The follower's constraint set contains the variable upper bound constraint $0 \leq y \leq u(1 - x)$
  - Together, we'll refer to these as *interdiction constraints*

This leads to the mixed-integer zero-sum static Stackelberg problem with interdiction constraints (MZSSPIC):

$$\begin{aligned} \min_x \quad & f(x, y) \\ \text{subject to} \quad & g(x, y) \leq 0 \\ & b(x, y) \leq B \\ & x \in \{0, 1\} \\ & y \in \operatorname{argmax} \{ f(x, y) : h(x, y) \leq 0, 0 \leq y \leq u(1 - x), y \in Y_{INT} \} \end{aligned} \tag{7}$$

Motivation

- In many applications, the leader may not be subject to a hard budget constraint
- Instead, it may be more helpful to analyze the tradeoff between resources spent and the resulting effect on the objective.
- This leads us to formulate this a bicriteria optimization problem

Moving the leader's budget constraint into the objective function via bicriteria programming yields the bicriteria mixed-integer zero-sum static Stackelberg problem with interdiction constraints (BMZSSPIC):

$$\begin{aligned} \text{vmin} \quad & [b(x,y), f(x,y)] \\ \text{subject to} \quad & g(x,y) \leq 0 \\ & x \in \{0,1\} \\ & y \in \operatorname{argmax} \{f(x,y) : h(x,y) \leq 0, 0 \leq y \leq u(1-x), y \in Y_{INT}\} \end{aligned} \tag{8}$$

The Bicriteria Integer Program

Consider the general bicriteria integer program (BIP):

$$\text{vmax}_{x \in X}[f_1(x), f_2(x)]. \tag{9}$$

We are looking for *efficient* solutions to (9).

Definition

A feasible solution $\hat{x} \in X$ is *efficient* if there is no other $x \in X$ such that

$$\begin{aligned} f_i(x) &\geq f_i(\hat{x}), \text{ for } i = 1, 2 \text{ and} \\ f_i(x) &> f_i(\hat{x}) \text{ for some } i. \end{aligned}$$

We say $\hat{x} \in X$ is *strongly efficient* if it is efficient and

$$f_i(\hat{x}) > f_i(x) \text{ for all } i.$$

Let X_E denote the set of efficient solutions and Y_E denote the image of X_E in the outcome space (i.e. $Y_E = f(X_E)$). Y_E is the set of Pareto outcomes.

Weighted Sums

We can convert (9) into a single-objective problem with a nonnegative linear combination of the objective functions (Geoffrion, 1968):

$$\max_{x \in X} \alpha f_1(x) + (1 - \alpha) f_2(x). \tag{10}$$

for $0 \leq \alpha \leq 1$. Solutions to (10) are

- In the Pareto set
- On the *convex upper envelope*
  - On the Pareto portion of the boundary of $\mathrm{conv}(Y)$
  - We call these outcomes *supported*

Comment

Not every Pareto outcome is supported.

WCN Algorithm

Ignoring some technical details, we can generate the entire Pareto set by solving

$$\min_{x \in X} \left\{ \|(f_1(x) - f_1(x_1^*)), (f_2(x) - f_2(x_2^*))\|_\infty^\beta \right\} \tag{11}$$

where $\|(f_1, f_2)\|_\infty^\beta = \max\{\beta|f_1|, (1-\beta)|f_2|\}$ and (x_1^*, x_2^*) is the *ideal point*, found by solving with respect to each objective function individually (Ralphs et al., 2004).

Applying standard techniques yields the equivalent program

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq \beta\,(f_1(x_1^*) - f_1(x)) \\ & z \geq (1-\beta)\,(f_2(x_2^*) - f_2(x)) \\ & x \in X \end{aligned} \tag{12}$$

Back to BMZSSPIC

Applying these results to BMZSSPIC yields the subproblem $P(\beta)$:

$$\begin{aligned} \max \quad & z \\ \text{subject to} \quad & z \geq \beta\left(b(x,y) - b(x_1^*, y_1^*)\right) \\ & z \geq (1-\beta)\left(f(x,y) - f(x_1^*, y_1^*)\right) \\ & g(x,y) \leq 0 \\ & x \in \{0,1\} \\ & y \in \operatorname{argmax}\{f(x,y) : h(x,y) \leq 0, \\ & \qquad 0 \leq y \leq u(1-x), \\ & \qquad y \in Y_{INT}\} \end{aligned} \tag{13}$$

for $0 \leq \beta \leq 1$.

Comment

$P(\beta)$ is a static Stackelberg game.

Some Notation

The following notations, definitions, and examples are taken from Moore and Bard (1990). Let:

$$
\begin{aligned}
\Omega &= \{(x,y) : g(x,y) \leq 0, x \in X_{INT}, h(x,y) \leq 0, y \in Y_{INT}\} \\
\Omega(X) &= \{x \in X : g(x,y) \leq 0 : \exists y \text{ such that } (x,y) \in \Omega\} \\
\Omega(x) &= \{y : h(x,y) \leq 0, y \in Y_{INT}\} \\
M(x) &= \{y : \text{argmax}(f(y') : y' \in \Omega(x))\} \\
\text{IR} &= \{(x,y) : x \in \Omega(X), y \in M(x)\}
\end{aligned}
$$

Definition

If $\bar{y} \in M(\bar{x})$ then $\bar{y}$ is said to be optimal with respect to $\bar{x}$; such a pair will be called *bilevel feasible*.

General Branch & Bound

General Fathoming Rules for Branch & Bound:

1. The relaxed suproblem has no feasible solution.
2. The solution of the relaxed subproblem is no greater than the value of the incumbent.
3. The solution of the relaxed subproblem is feasible to the original problem.

Comment

Only Rule 1 holds for $P(\beta)$!

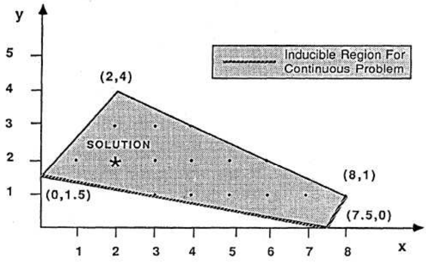
Example 1

Consider the mixed-integer BLP:

$$
\begin{aligned}
\max_{x \in \mathbb{Z}^+} \quad & F(x,y) = x + 10y \\
\text{subject to} \quad & y \in \operatorname{argmax} \{ f(x,y) = -y : -25x + 20y \leq 30 \\
& \qquad\qquad\qquad\qquad x + 2y \leq 10 \\
& \qquad\qquad\qquad\qquad 2x - y \leq 15 \\
& \qquad\qquad\qquad\qquad 2x + 10y \geq 15 \\
& \qquad\qquad\qquad\qquad y \in \mathbb{Z}^+ \} .
\end{aligned}
$$

Example 1 (cont)

Here we can see Ω:



From this example, we have the following observations:

1. The solution of the relaxed problem does not give a valid bound on the solution of the original problem.
2. Solutions to the relaxed problem that are in the inducible region cannot necessarily be fathomed.

Example 1

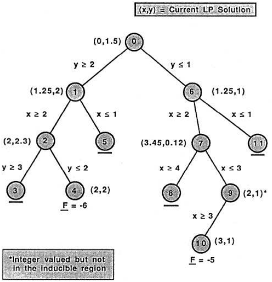
Consider the mixed-integer BLP:

$$
\begin{aligned}
\max_{x \in \mathbb{Z}^+} \quad & F(x,y) = -x - 2y \\
\text{subject to} \quad & y \in \operatorname{argmax} \{ f(x,y) = y : -x + 2.5y \leq 3.75 \\
& \qquad\qquad x + 2.5y \geq 3.75 \\
& \qquad\qquad 2.5x + y \leq 8.75 \\
& \qquad\qquad y \in \mathbb{Z}^+ \} .
\end{aligned}
$$

We can check that the constraint region contains the 3 integer points $(2,1), (2,2), (3,2)$, with the optimal solution $(x^*, y^*) = (3,1)$ and $F = -5$.

Example 2 (cont)

Here is a branch and bound tree that could result from a typical branch and bound scheme:



Consider node 9, with solution $(x, y) = (2, 1)$ with $F = -4$.

Example 2 (cont)

It is easy to check that

$$\begin{aligned}(2,1) &\in \Omega \\ (2,1) &\in IR.\end{aligned}$$

But, even though $(2,1)$ is integer, it cannot be fathomed because it is not **bilevel feasible**. To see this, note that if the leader chooses $x = 2$, the follower's optimal response is $y = 2$. This leads to the following observation:

1. All integer solutions to the relaxed BLP with some of the follower's variables restricted cannot, in general, be fathomed.

Fixing Rule 2

Let

- H_k^L and H_k^F denote the sets of bounds place on the integer variables controlled by the leader and follower, respectively
- $H_k^F(0, \infty)$ indicate that no bounds have been placed on the follower's integer variables, other than those in the original problem
- The *high point* solution be defined as the solution to (continuous) subproblem k when the follower's objective is removed.

Theorem (Moore and Bard (1990))

Given H_k^L and $H_k^F(0, \infty)$ and the high point solution (x^k, y^k), $F_k^H = F(x^k, y^k)$ is an upper bound on the solution of the mixed integer BLP at node k.

The high point solution at node k can be used as a bound to determine if the subproblem can be fathomed if, once the leader has made a decision, the follower can optimize without any *a priori* or *artificial* restrictions.

Fixing Rule 2 (cont)

If we have placed restrictions on some of the follower's variables, we can still use the high point solution as an upper bound, under the conditions of Theorem 2.

- Let $\alpha_j^k > 0$ or $\beta_j^k < U_j$ be lower and upper bounds placed on the jth integer variable controlled by the follower at subproblem k.

Theorem (Moore and Bard (1990))

Given H_k^L and H_k^F and the high point solution (x^k, y^k), $F_k^H = F(x^k, y^k)$ is an upper bound on the solution of the mixed integer BLP defined by the current path in the tree if none of the follower's restricted integer variables are at either $\alpha_j^k > 0$ or $\beta_j^k < U_j$.

Fixing Rule 2 (cont)

The condition of Theorem 2 is quite strong. The following corollary provides some help:

Corollary (Moore and Bard (1990))

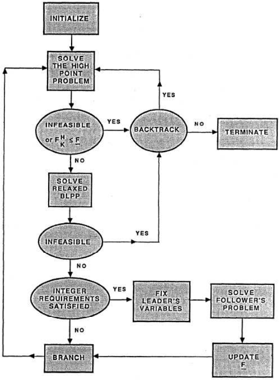
Given H_k^L and H_k^F, let (x^k, y^k) be the high point solution of the relaxed BLP with the bounds in H_k^F relaxed. Then, $F_k^H = F(x^k, y^k)$ is an upper bound on the solution of the mixed integer BLP defined by the current path in the tree.

This is still a fairly restrictive result. This is mainly due to the following observation:

- In the BLP, once the leader makes a decision, the follower is free to act without regard to restrictions placed on the leader's variables earlier in the tree. This is a sharp contrast to MIP, where those bounds are valid.

Modified Branch & Bound Algorithm

Below is the flow diagram of the modified depth-first branch and bound algorithm suggested by Moore and Bard (1990):



Solving the Relaxed BLP

Comment

In the relaxed BLP, the subproblem is an LP, so we can replace the objective with KKT conditions.

Taking this approach yields a nonconvex NLP. Two main approaches have been taken to solve this problem:

1. Linearize complementary slackness constraints by introducing binary variables and solve the 0-1 program with a MIP solver (Fortuny-Amat and McCarl, 1981).
2. Relax the complementary slackness conditions and branch on KKT multipliers, checking the complementary slackness conditions at each iteration (Bard and Moore, 1990).

Future Directions

The following future directions are planned:

1. Develop a framework that solves BMZSSPIC, using a more general branch and bound scheme than a standard MIP solver
2. Consider different approaches to solving the relaxed BLP (i.e. cutting plane techniques)
3. Better understand where BMZSSPIC fits into the mathematical universe

References

Bard, J. and J. Moore 1990. A branch and bound algorithm for the bilevel programming problem. *SIAM Journal on Scientific and Statistical Computing* **11**(2), 281–292.

Ben-Ayed, O. and C. Blair 1990. Computational difficulties of bilevel linear programming. *Operations Research* **38**, 556–560.

Calamai, P. and L. Vincente 1994. Generating quadratic bilevel programming problems. *ACM Transactions on Mathematical Software* **20**, 103–119.

Fortuny-Amat, J. and B. McCarl 1981. A representation and economic interpretation of a two-level programming problem. *Journal of the Operations Research Society* **32**, 783–792.

Geoffrion, A. 1968. Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications* **22**, 618–630.

Hansen, P., B. Jaumard, and G. Savard 1992. New branch-and-bound rules for linear bilevel programming. *SIAM Journal on Scientific and Statistical Computing* **13**(5), 1194–1217.

Jeroslow, R. 1985. The polynomial hierarchy and a simple model for competitive analysis. *Mathematical Programming* **32**, 146–164.

Moore, J. and J. Bard 1990. The mixed integer linear bilevel programming problem. *Operations Research* **38**(5), 911–921.

Ralphs, T., M. Saltzman, and M. Wiecek 2004. An improved algorithm for biobjective integer programming and its application to network routing problems. Technical Report 04T-004, Lehigh University Industrial and Systems Engineering.