A non-standart approach to Duality

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Outline

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   - Duality for Integration
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   - Duality for Summation

2. Discrete Farkas Lemma
   - Continuous Case
   - Discrete Case
   - LP formulation
   - Superadditive Dual

References:

- Jean B. Lasserre, *Duality and a Farkas lemma for integer programs*. in: Optimization: Structure and Applications (E. Hunt and C.E.M. Pearce, Editors), Kluwer Academic Publishers (2004)
- Jean B. Lasserre, *Integer programming duality and superadditive functions*. Contemporary Mathematics 374, pp. 139–150. (2005)

Formulation

LP | **IP**

$$\mathbb{P}: \quad f(b,c) := \max_{x\in\Omega(b)} \; c'x \quad \leftrightarrow \quad \mathbb{P}_d: \quad f_d(b,c) := \max_{x\in\Theta(b)} \; c'x$$

$$\updownarrow \qquad\qquad \updownarrow$$

Integration | **Summation**

$$\mathbb{I}: \quad \hat{f}(b,c) := \int_{\Omega(b)} e^{c'x} ds \quad \leftrightarrow \quad \mathbb{I}_d: \quad \hat{f}_d(b,c) := \sum_{x\in\Theta(b)} e^{c'x}$$

where $\Omega(b) := \{x \in \mathbb{R}^n : Ax = b, \; x \geq 0\}$ and $\Theta(b) := \Omega(b) \cap \mathbb{Z}^n$.

The simple relation is that:

$$e^{f(b,c)} = \lim_{r\to\infty} \hat{f}(b,rc)^{1/r}, \quad e^{f_d(b,c)} = \lim_{r\to\infty} \hat{f}_d(b,rc)^{1/r}$$

or equivalently

$$f(b,c) = \lim_{r\to\infty} \frac{1}{r} \ln \hat{f}(b,rc), \quad f_d(b,c) = \lim_{r\to\infty} \frac{1}{r} \ln \hat{f}_d(b,rc)$$

Let $\mathbb{P}^*, \mathbb{P}_d^*$ be some dual problems of $\mathbb{P}, \mathbb{P}_d$.

- The well-known strong dual $\mathbb{P}^*$ can be obtained by *Legendre-Fenchel Duality* formulation. Let $f : \mathbb{R}^n \to \mathbb{R}$, then the *Fenchel conjugate*:

$$f^*(y) := \sup_{x \in \mathbb{R}^n} \{x'y - f(x)\} \ \ \forall y \in \mathbb{R}^n$$

  If f is convex and lower semi-continuous $\Leftrightarrow \ \ f := f^{**}$.
  Then, noting that $f(y, c)$ is concave,

$$\mathbb{P}^* : f^{**}(b, c) := \inf_{\lambda \in \mathbb{R}^m} \{\lambda' b - f^*(\lambda, c)\} = \min_{\lambda \in \mathbb{R}^m} \{b'\lambda \mid A'\lambda \geq c\}$$

- No available transformation to get a strong $\mathbb{P}_d^*$ (except the subadditive formulation, see my previous talk...)

Let $\mathbb{I}^*, \mathbb{I}_d^*$ be the similar transforms of $\mathbb{I}, \mathbb{I}_d$ using *Laplace-transform* and $\mathbb{Z}$*-transform*, respectively.

Lasserre calls $\mathbb{I}^*, \mathbb{I}_d^*$ the *natural duals*: we can get closed form equations for $\hat{f}(b, c)$ and $\hat{f}_d(b, c)$.

Laplace transform $f^* : \mathbb{C}^m_+ \to \mathbb{C}$ of $f : \mathbb{R}^m_+ \to \mathbb{R}$ is

$$f^*(\lambda) := \int_{\mathbb{R}^n_+} e^{-\lambda' y} f(y) dy$$

Applying to $\hat{f}(b, c)$:

$$\begin{aligned}\hat{F}(\lambda, c) &:= \int_{\mathbb{R}^m_+} e^{-\lambda' y} \hat{f}(y, c) dy \\ &:= \prod_{k=1}^{n} \frac{1}{(A'\lambda - c)_k}\end{aligned}$$

Note that, $\hat{F}(\lambda, c)$ is well defined when $\Re(A'\lambda - c) > 0$. To get $\mathbb{I}^*$ and hence a closed form $\hat{f}(b, c)$, we solve the *inverse* Laplace transform problem:

$$\begin{aligned} \mathbb{I}^* : \hat{f}(b,c) &:= \frac{1}{(2i\pi)^m} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{-\lambda' b} \hat{F}(\lambda, c) d\lambda \\ &= \frac{1}{(2i\pi)^m} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{-\lambda' b}}{\prod_{k=1}^{n} (A'\lambda - c)_k} d\lambda \end{aligned}$$

where γ is fixed and satisfies $(A'\gamma - c) > 0$. This complex integral can be solved directly by Cauchy residual techniques.

In fact, the multidimensional poles of $\hat{F}$ are real and solve $p_B A_B = c_B$ for all bases of $\Omega(b)$. Then:

$$\hat{f}(b,c) = \sum_{x \in b.f.s \text{ of } \Omega(b)} \frac{e^{c'x}}{det(B) \prod_{k \in NB} (-c_k + p^B A_k)}$$

where B is the basis of the corresponding b.f.s.
In other words, $\hat{f}(b,c)$ is a weighted summation over the vertices of $\Omega(b)$.

$\mathbb{Z}$-transform $f^* : \mathbb{C}_+^m \to \mathbb{C}$ of $f : \mathbb{Z}_+^m \to \mathbb{R}$ is

$$f^*(z) := \sum_{y \in \mathbb{Z}_+^m} z^{-y} f(y)$$

Applying to $\hat{f}_d(b, c)$:

$$\hat{F}_d(z, c) \quad := \quad \prod_{k=1}^{n} \frac{1}{1 - e^{c_k} z^{-A_k}}$$

which is well-defined if $|z^{A_k}| > e^{c_k} \quad k = 1, ..., n$

To get $\mathbb{I}_d^*$ and hence a closed form $\hat{f}_d(b,c)$, we solve the *inverse* $\mathbb{Z}$-transform problem:

$$\mathbb{I}_d^* : \hat{f}_d(b,c) \quad := \quad \frac{1}{(2i\pi)^m} \int_{|z|=\gamma} \hat{F}_d(z,c) z^{b-\mathbf{1}} dz$$

where γ satisfies $\gamma^{A_k} > e^{c_k} \quad k = 1, ..., n$.
Cauchy residue technique can be used, however, we have *complex* poles!

Bases of $\omega(b)$ provide these poles. Each basis B provides $det(B)$ complex poles in the form of:

$$z(k) = e^{p_B + 2i\pi \frac{v}{det(B)}} \quad for \quad k = 1, ..., det(B)$$

where $v \in \{v \in \mathbb{Z}^m | v'B = 0 \quad mod \ det(B)\}$

Combining these poles with discrete Brian and Vergne's formula:

$$\hat{f}_d(b,c) = \sum_{x \in b.f.s \ of \ \Omega(b)} e^{c'x} \times U_B(b,c)$$

Then,

$$\begin{aligned} f_d(b,c) &= \lim_{r\to\infty} \frac{1}{r} \ln \hat{f}_d(b, rc) \\ &= \max_{x \in b.f.s \ of \ \Omega(b)} \{c'x + \lim_{r\to\infty} \frac{1}{r} \ln U_B(b, rc)\} \\ &= \max_{x \in b.f.s \ of \ \Omega(b)} \{c'x + \frac{1}{q}(deg(P_b) - deg(Q_b))\} \\ &= c'x^* + \rho \end{aligned}$$

where q is the l.c.m of $det(B)$: B is a feasible basis , P_b and Q_b are some real valued polynomials.

Note that, ρ is the value of the Gomory group/asymptotic problem!

Continuous Case

Note that a polynomial $Q \subset \mathbb{R}[\lambda_1, ..., \lambda_m]$ can be written

$$Q(\lambda) = \sum_{\alpha \in S} Q^\alpha \lambda^\alpha = \sum_{\alpha \in S} Q^\alpha \lambda_1^{\alpha_1} ... \lambda_m^{\alpha_m}$$

where $S \subset \mathbb{N}^m$ and Q^α are real coefficients $\forall \alpha \in S$.

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Then the folllowing two statements are equivalent:
(a) The linear system $Ax = b$ has a nonnegative solution $x \in \mathbb{R}^n$.
(b) The polynomial $b'\lambda$ can be written

$$b'\lambda = \sum_{j=1}^{n} Q_j(\lambda)(A'_j\lambda), \ \lambda \in \mathbb{R}^m$$

for some polynomials $Q_j \subset \mathbb{R}[\lambda_1, ..., \lambda_m], j = 1, ..., n$, all with nonnegative coefficients.

Proof:

Discrete Case

Let, wlog (as long as the feasible region is compact), $A \in \mathbb{N}^{m\times n}$, $b \in \mathbb{N}^m$.
Then the folllowing two statements are equivalent:
(a) The linear system $Ax = b$ has a solution $x \in \mathbb{N}^n$.
(b) The polynomial $z^b - 1$ can be written

$$z^b - 1 = \sum_{j=1}^{n} Q_j(z)(z^{A_j} - 1), \;\; z \in \mathbb{R}^m$$

for some polynomials $Q_j \subset \mathbb{R}[z_1, ..., z_m]$, $j = 1, ..., n$, all with nonnegative coefficients. Moreover, the total degree of each polynomial Q_j can be bounded by $b^* = \sum_{j=1}^m b_j - \min_{k=1,\ldots,n} \sum_{i=1}^{m} A_{ik}$.

Proof:

Discrete Case

Let $q \geq 0$ be the vector of nonnegative coefficients of all polynomials Q_j's. If

$$D := \{Mq = r, \quad q \geq 0\} \neq \emptyset$$

then, there exist such polynomials. Note that $Mq = r$ state that the polynomials $z^b - 1$ and $\sum_{j=1}^{n} Q_j(z)(z^{A_j} - 1)$ are identical by equating the respective coefficients.

Discrete Case

- Each Q_j may be restricted to contain only monomials

$$\{z^\alpha : \alpha \le b - A_j, \ \ \alpha \in \mathbb{N}^m\}$$

- Hence, in D, $q \in \mathbb{R}^s$ and $M \in \mathbb{R}^{p \times s}$ where

$$\begin{aligned} p &= \prod_{i=1}^{m} (b_i + 1) \\ s &= \sum_{j=1}^{n} s_j \quad \textit{with} \quad s_j = \prod_{i=1}^{m} (b_i - A_{ij} + 1), \quad j = 1, \ldots, n \end{aligned}$$

  Note that,
  p is the number of monomials z^α with $\alpha \le b$ and
  s_j is the number of monomials z^α with $\alpha - A_j \le b$.

- ***Example:***
- Note that M is totally unimodular.

An LP equivalent to $\mathbb{P}_d$

Let $e_{s_j} = (1, ..., 1) \in \mathbb{R}^{s_j}, \ \ j = 1, ..., n$ and let $E \in \mathbb{N}^{n \times s}$ be the n-block diagonal matrix, whose each diagonal block is a row vector e_{s_j}. Then,

1. $f_d(b, c) = max\{c' Eq \mid Mq = r \ , \ \ q \geq 0\}$
2. If q^* is an optimal solution, then $x^* := Eq^*$ is the associated optimal solution of $\mathbb{P}_d$.

Proof:

A class of superadditive functions

Let

- $\mathcal{D} \subset \mathbb{N}^m$ be a finite set s.t 0 $\in \mathcal{D}$ and if $\alpha \in \mathcal{D} \Rightarrow \beta \in \mathcal{D}, \ \forall \beta \leq \alpha$.
- $\Delta_{\mathcal{D}}$ be the set of functions $\pi : \mathbb{N}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ s.t. $\pi(0) = 0$ and $\pi(\alpha) < +\infty$ *if* $\alpha \in \mathcal{D}$, or $\pi(\alpha) = +\infty$, otherwise.
- Given $\pi \in \Delta_{\mathcal{D}}$, $f_\pi : \mathbb{N}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as

$$f_\pi(x) := \inf_{\alpha \in \mathcal{D}} \{\pi(\alpha + x) - \pi(\alpha)\}, \quad x \in \mathbb{N}^m$$

Lemma: For every $\pi \in \Delta_{\mathcal{D}}$:

1. $f_\pi \in \Delta_{\mathcal{D}}$
2. $f_\pi \leq \pi$, and f_π is superadditive
3. if π is superadditive, then $\pi = f_\pi$.

Proof:

Superadditive Dual

Let $\mathcal{D} := \{\alpha \in \mathbb{N}^m \mid \alpha \leq b\}$. Then the dual of $max\{c'Eq \mid Mq = r\ ,\quad q \geq 0\}$ is

$$\begin{aligned} \min_{\gamma} \quad & \gamma(b) - \gamma(0) \\ s.t \quad & \gamma(\alpha + A_j) - \gamma(\alpha) \geq c_j, \quad \alpha + A_j \in \mathcal{D}, \quad j = 1, ..., n \end{aligned}$$

Letting $\pi(\alpha) = \gamma(\alpha) - \gamma(0),\ \forall \alpha \in \mathcal{D}$ and extending π to $\mathbb{N}^m$, the dual becomes

$$\begin{aligned} \rho_1 = \min_{\pi \in \Delta_{\mathcal{D}}} \quad & \pi(b) \\ s.t \quad & \pi(\alpha + A_j) - \pi(\alpha) \geq c_j, \quad \alpha \in \mathcal{D}, \quad j = 1, ..., n \end{aligned}$$

Superadditive Dual

Now, assume that $f_d(b, c) > -\infty$ and consider the following problem

$$\rho_2 = \inf_{\pi \in \Delta_{\mathcal{D}}} \quad f_\pi(b)$$
$$s.t \quad f_\pi(A_j) \geq c_j, \quad j = 1, ..., n$$

where $f_\pi : \mathbb{N}^m \to \mathbb{R}$ defined as before for every $\pi \in \Delta_{\mathcal{D}}$. Then,

$$f_d(b, c) = \rho_1 = \rho_2 = f_{\pi^*}(b) \;\; for \;\; some \;\; \pi^* \in \Delta_{\mathcal{D}}$$

Proof:
Example:

Superadditive Dual

Note the similar dual formulation of Wolsey

$$\begin{aligned} \min_{\pi} \quad & \pi(b) \\ s.t \quad & \pi(\lambda) + \pi(\mu) \leq \pi(\lambda + \mu), \;\; 0 \leq \lambda + \mu \leq b \\ & \pi(A_j) \geq c_j, \;\; j = 1, ..., n \\ & \pi(0) = 0 \end{aligned}$$

where the first constraint set state that $\pi : \mathcal{D} \rightarrow \mathbb{R}$ is superadditive. The number of variables are the same, however, this one has $\mathcal{O}(p^2)$ constraints whereas the introduced one has $\mathcal{O}(np)$.