# Primal Heuristics

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# Outline

# The Feasibility Pump

Matteo Fischetti, Fred Glover and Andrea Lodi, *The Feasibility Pump*, Math. Programming. Ser A (March 2005)

# Introduction

- Finding a feasible solution for MIP is NP-Hard and is difficult in practice for some important models and applications.
- Other successful primal heuristics like RINS and Local Branching can be used only if an initial feasible solution is known.
- The earlier a feasible solution is found, the better...

- We consider a generic MIP of the following form:

$$\begin{aligned} min \quad & c^T x \\ Ax & \geq b \\ x_j \quad \text{integer} & \qquad \forall j \in \mathbb{I} \end{aligned}$$

- The authors intend to find a feasible solution for the problem.

- Let $P := \{x : Ax \geq b\}$ denote the polyhedron associated with the LP relaxation of the given MIP.
- The rounding $\tilde{x}$ of x is obtained by setting $\tilde{x}_j := [x_j]$ if $j \in \mathbb{I}$ and $\tilde{x}_j := x_j$ otherwise.
- The $L_1$-norm distance between a generic point $x \in P$ and a given integer point $\tilde{x}$ is defined as:

$$\Delta(x, \tilde{x}) = \sum_{j \in \mathbb{I}} |x_j - \tilde{x}_j|$$

- The distance function may then be written as:

$$\Delta(x,\tilde{x}) = \sum_{j\in\mathbb{I}:\tilde{x}_j=l_j}(x_j-l_j)+\sum_{j\in\mathbb{I}:\tilde{x}_j=u_j}(u_j-x_j)+\sum_{j\in\mathbb{I}:l_j<\tilde{x}_j<u_j}(x_j^+ + x_j^-)$$

with

$$x_j = \tilde{x}_j + x_j^+ - x_j^-, \quad x_j^+, x_j^- \geq 0, \forall j \in I : l_j < \tilde{x}_j < u_j$$

- for mixed 0-1 programs, the additional variables are not needed.
- Given an integer point $\tilde{x}$, the closest point $x^* \in P$ can be determined by solving the LP

$$\min\{\Delta(x,\tilde{x}) : Ax \geq b\}$$

# The basic scheme

- We start with any $x^* \in P$ and define its rounding $\tilde{x}$
- At each iterationm we look for a point $x^* \in P$ which is closest to the current $\tilde{x}$ by solving the problem:

  $$min\{\Delta(x, \tilde{x}) : x \in P\}$$

  Assuming the distance function is chosen appropriately, this is an easily solvable LP program.
- If $\Delta(x, \tilde{x}) = 0$, then $x^*$ is a feasible MIP solution and we are done.
- Otherwise, we replace $\tilde{x}$ with the rounding of $x^*$ and repeat.

The basic algorithm is stated as follows:

- $x^* := argmin\{c^T x : Ax \geq b\}$;
- if $x^*$ is integer, return $x^*$;
- Let $\tilde{x} := [x^*]$ (rounding of $x^*$)
- while (time < TL ) do
- compute $x^* := argmin\{\Delta(x, \tilde{x}) : Ax \geq b\}$;
- if $x^*$ is integer, return $x^*$;
- if $\exists j \in \mathbb{I} : [x_j^*] \neq \tilde{x}_j$ then
- $\tilde{x} := [x^*]$
- else
- flip the TT=rand(T/2, 3T/2) entries $\tilde{x}_j (j \in \mathbb{I})$ with highest $|x_j^* - \tilde{x}_j|$
- endif
- enddo

- The flipping deals with stalling issues..
- Possibility of Cycling.. A pertubation mechanism is needed
- On detecting a cycle, for each $j \in \mathbb{I}$ generate a random value $\rho_j \in [-0.3, 0.7]$ and flip $\tilde{x}_j$ in case $|x_j^* - \tilde{x}_j| + max[\rho_j, 0] > 0.5$
- FP generates two (hopefully convergent) trajectories of points $x^*$ and $\tilde{x}$ that satisfies feasibility in a complementary but partial way...

- Improving computing time
- Improving the quality of the heursitic solution.
- FP is also applicable to MINLP problems.

# References

*OCTANE: A new heuristic for pure 0-1 programs*, Egon Balas, Sebastian Ceria, Milind Dawande, Francis Margot, Gabor Pataki. 2001. Operations Research. Vol. 49, No. 2, pages 207-225.

# Notation

- IP:

$$
\begin{aligned}
&\text{Min } cx\,,\\
&\text{s.t. } Ax \geq b\\
&x_i \in \{0,1\}, (i = 1, \ldots, n)
\end{aligned}
$$

- Let:

$$
\begin{aligned}
K &= \left\{x \in \mathbb{R}^n : -\frac{e}{2} \leq x \leq \frac{e}{2}\right\},\\
K^* &= \left\{x \in \mathbb{R}^n : ||x||_1 \leq \frac{1}{2}n\right\}\\
&= \left\{x \in \mathbb{R}^n : \delta x \leq \frac{1}{2}n, \forall \delta \in \{\pm 1\}^n\right\}
\end{aligned}
$$

- What are the extreme points of $K$?
- What are the facets of $K^*$?
- What is relation between $K + \frac{1}{2}e$ and $K^* + \frac{1}{2}e$?
- $\frac{1}{2}\delta + \frac{1}{2}e = x$

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- Geometry of intersection cuts.
- Deepening the cuts
- What happens if degenerate?
- How is it a primal heuristic?

- Let $\bar{x} = x - \frac{1}{2}e$.
- $r = \bar{x} + \lambda a, \lambda \geq 0$ is a half line originating at $\bar{x}$.
- Let $\Lambda(\delta)$ be the distance of $\bar{x}$ from $\delta$.
- Then:

$$
\begin{aligned}
(\bar{x} + \Lambda(\delta)a)\delta &= \frac{1}{2}n \\
\Rightarrow \quad \Lambda(\delta) &= \frac{\frac{n}{2} - \delta\bar{x}}{\delta a} > 0
\end{aligned}
$$

Let $I \subset \mathbb{N}$, and

$$p(\delta, I) = -\sum_{i \in I} \delta_i \bar{x}_i, \qquad P(\delta) = \frac{n}{2} + p(\delta, \mathbb{N})$$

$$q(\delta, I) = \sum_{i \in I} \delta_i a_i, \qquad Q(\delta) = q(\delta, \mathbb{N})$$

$$\lambda(\delta, I) = p(\delta, I)/q(\delta, I)$$

$$v(i) = -\frac{\bar{x}_i}{a_i}(i = 1, 2, \ldots, n)$$

Note that $\Lambda(\delta) = \frac{P(\delta)}{Q(\delta)}$

Decreasing Flip:
$i \in N$ is a decreasing flip if and only if one of the two conditions hold:

1. $\delta_i = 1$ and $v(i) > \Lambda(\delta)$
2. $\delta_i = -1$ and $v(i) < \Lambda(\delta)$

$\delta$ is first reachable if and only if there is no decreasing single flip for $\delta$.

After some magical algebraic deductions ...

Algorithm First Facet

```
Let δ be a reachable facet of K*.
while (there is a decreasing flip i for δ)
   set δ = δ◊i.
end while
end
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