

# *Branching on general disjunctions:*

## *Complexity and Computational Experiments*

Ashutosh Mahajan<sup>1</sup>, Ted Ralphs<sup>1</sup>

<sup>1</sup>Cor@l Lab

Department of Industrial and Systems Engineering  
Lehigh University

Informs 2008



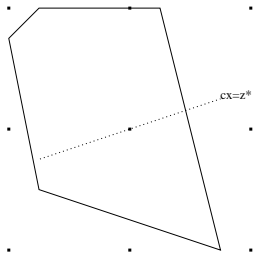
# Branch and Bound Procedure

$$\text{MIP: } \min_{x \in S} cx$$

$$S: \begin{array}{l} Ax \geq b \\ x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

$$\text{LP: } \min_{x \in \mathcal{P}} cx$$

$$\mathcal{P}: \begin{array}{l} Ax \geq b \\ x \in \mathbb{R}^n, \end{array}$$



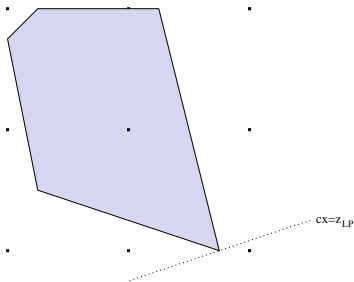
# Branch and Bound Procedure

$$\text{MIP: } \min_{x \in S} cx$$

$$S: \begin{array}{l} Ax \geq b \\ x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

$$\text{LP: } \min_{x \in \mathcal{P}} cx$$

$$\mathcal{P}: \begin{array}{l} Ax \geq b \\ x \in \mathbb{R}^n, \end{array}$$



$z_{LP}$

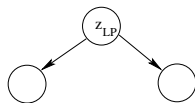
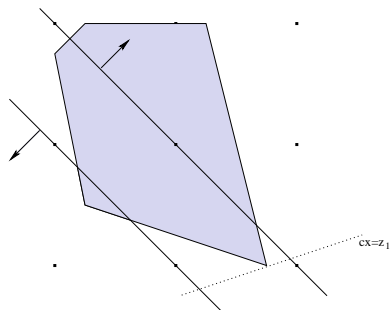
# Branch and Bound Procedure

$$\text{MIP: } \min_{x \in S} cx$$

$$S: \begin{array}{l} Ax \geq b \\ x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

$$\text{LP: } \min_{x \in \mathcal{P}} cx$$

$$\mathcal{P}: \begin{array}{l} Ax \geq b \\ x \in \mathbb{R}^n, \end{array}$$



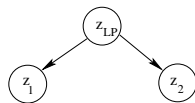
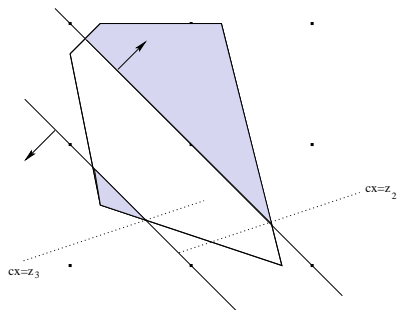
# Branch and Bound Procedure

$$\text{MIP: } \min_{x \in S} cx$$

$$S: \begin{array}{l} Ax \geq b \\ x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

$$\text{LP: } \min_{x \in \mathcal{P}} cx$$

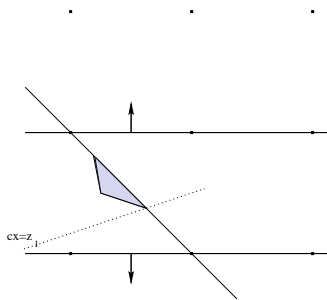
$$\mathcal{P}: \begin{array}{l} Ax \geq b \\ x \in \mathbb{R}^n, \end{array}$$



# Branch and Bound Procedure

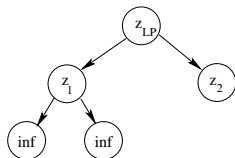
$$\text{MIP: } \min_{x \in S} cx$$

$$S: \begin{array}{l} Ax \geq b \\ x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$



$$\text{LP: } \min_{x \in \mathcal{P}} cx$$

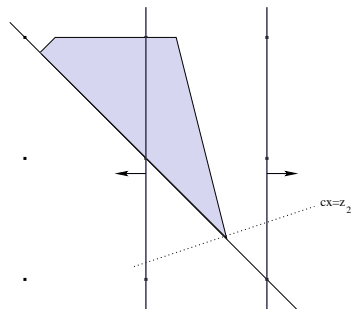
$$\mathcal{P}: \begin{array}{l} Ax \geq b \\ x \in \mathbb{R}^n, \end{array}$$



# Branch and Bound Procedure

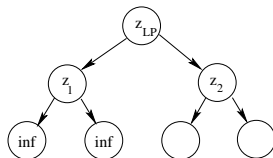
$$\text{MIP: } \min_{x \in S} cx$$

$$S: \begin{array}{l} Ax \geq b \\ x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$



$$\text{LP: } \min_{x \in \mathcal{P}} cx$$

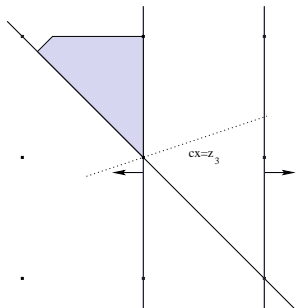
$$\mathcal{P}: \begin{array}{l} Ax \geq b \\ x \in \mathbb{R}^n, \end{array}$$



# Branch and Bound Procedure

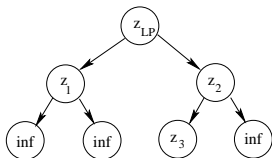
$$\text{MIP: } \min_{x \in S} cx$$

$$S: \begin{array}{l} Ax \geq b \\ x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

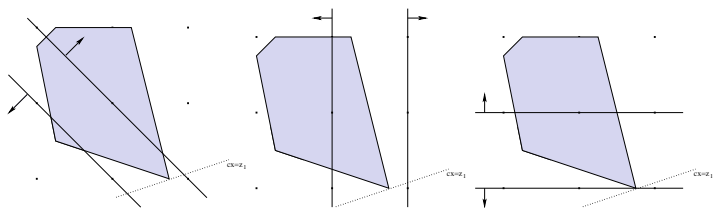


$$\text{LP: } \min_{x \in \mathcal{P}} cx$$

$$\mathcal{P}: \begin{array}{l} Ax \geq b \\ x \in \mathbb{R}^n, \end{array}$$



## Branching Decision



- ▶ Any subsets  $S_1, S_2$  such that  $S \subseteq (\mathcal{P} \cup S_1) \cup (\mathcal{P} \cap S_2)$  could be used to partition
- ▶ We restrict ourselves to partitions that are polyhedra
- ▶ Assume that we only create at most two partitions in each iteration
- ▶ The branching rule  $x_i \leq \pi_0 \vee x_i \geq \pi_0 + 1 \quad i = 1, \dots, d$  is valid. (Variable disjunction)
- ▶ A more general valid disjunction is  $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$ , where  $(\pi, \pi_0) \in \mathbb{Z}^d \times \mathbb{0}^{n-d} \times \mathbb{Z}$ . (General disjunction)
- ▶ e.g.  $2x_1 + x_2 \leq 4 \vee 2x_1 + x_2 \geq 5$

## *Previous Work*

### Branching on Variables:

- ▶ Finding the minimum size tree is  $\mathcal{NP}$ -Hard. (Liberatore, 2000)
- ▶ Heuristics for improving bounds: **Strong branching**, Pseudo-cost branching, Reliability branching (Experiments by Linderoth and Savelsbergh, 1999, Achterberg et al., 2005)
- ▶ Heuristics for feasible solutions (Patel and Chinneck, 2007)
- ▶ Heuristics for conflict analysis and resolution (Chvátal, 1997, Achterberg, 2007)
- ▶ Heuristics based on thin directions (Derpich and Vera, 2006)
- ▶ Information theoretic approaches (Gilpin and Sandholm, 2007)

### Branching on General Disjunctions:

- ▶ Trees polynomial in size, when dimension is fixed (Lenstra 1983, other work by Aardal, Lovasz, Lenstra). Minimum size trees?
- ▶ Greedy heuristic by Owen and Mehrotra, 2001
- ▶ Local branching heuristic for feasibility (Fischetti and Lodi, 2003)
- ▶ Heuristics based on Gomory Cuts (Karamanov and Cornuéjols, 2007, Cornuéjols, 2008)

## Branching on General Disjunctions

$$\begin{array}{ll} \text{MIP:} & z^* = \min cx \\ & \text{s.t. } Ax \geq b \\ & x \in \mathbb{Z}^n \end{array}$$

$$\begin{array}{ll} \text{LP:} & z_{LP}^* = \min cx \\ & \text{s.t. } Ax \geq b \\ & x \in \mathbb{R}^n \end{array}$$

$$\begin{array}{ll} z_L^* = \min cx & \\ \text{s.t. } Ax \geq b & \\ \pi x \leq \pi_0 & \\ x \in \mathbb{R}^n & \end{array}$$

$$\begin{array}{ll} z_R^* = \min cx & \\ \text{s.t. } Ax \geq b & \\ \pi x \geq \pi_0 + 1 & \\ x \in \mathbb{R}^n & \end{array}$$

New lower bound,  $z_l = \min\{z_L^*, z_R^*\}$

### Objective

Find  $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$  such that  $z_l = \min\{z_L^*, z_R^*\}$  is maximized

# Complexity

## Optimization Problem

Find  $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$  such that  $z_l = \min\{z_L^*, z_R^*\}$  is maximized

- ▶ For branching on variables, this can be solved in polynomial-time
- ▶ Is there a similar result when considering general disjunctions?
- ▶ When  $(\pi, \pi_0) \in \{0, 1\}^{n+1}$ ?

## Decision Problem

Given a MIP and  $\hat{z}_l$ , does there exist  $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$  such that  $z_l = \min\{z_L^*, z_R^*\} > \hat{z}_l$

## Decision Problem (2)

Given a MIP, does there exist  $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$  such that LP relaxation of each branch is infeasible.

## Complexity Results

### Decision Problem (2)

Given a MIP, does there exist  $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$  such that LP relaxation of each branch is infeasible.

Above problem is:

- ▶  $\mathcal{NP}$ -complete in general (Reduction from Number Partitioning Problem)
- ▶  $\mathcal{NP}$ -complete when  $\pi \in \{0, 1\}^n$
- ▶  $\mathcal{NP}$ -complete when  $\pi \in \{-1, 0, 1\}^n$
- ▶  $\mathcal{NP}$ -complete when  $(\pi, \pi_0) \in \{0, 1\}^{n+1}$
- ▶  $\mathcal{NP}$ -complete when  $x \in \{0, 1\}^n$  and either of above four conditions hold (Reduction from 1-IN-3SAT)

### Optimization Problem

Find  $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$  such that  $z_l = \min\{z_L^*, z_R^*\}$  is maximized.

$\mathcal{NP}$ -hard

## Problem formulation: Maximum bound improvement

$$\begin{aligned} z_L^* &= \min cx \\ \text{s.t. } Ax &\geq b \\ \pi x &\leq \pi_0 \\ x &\in \mathbb{R}^n \end{aligned}$$

$$\begin{aligned} z_R^* &= \min cx \\ \text{s.t. } Ax &\geq b \\ \pi x &\geq \pi_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

### Decision Problem

Given a MIP and  $\hat{z}_l$ , does there exist  $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$  such that  $z_l = \min\{z_L^*, z_R^*\} > \hat{z}_l$ ?

Suppose  $(\hat{\pi}, \hat{\pi}_0)$  is one such disjunction. Then the following systems must be infeasible

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned}$$

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

## *Problem formulation: Maximum bound improvement*

If  $(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction,

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned}$$

&

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

must be  
infeasible

## *Problem formulation: Maximum bound improvement*

If  $(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction,

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned}$$

&

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

must be  
infeasible

$$\begin{aligned} z_1^* &= \min \hat{\pi}x \\ Ax &\geq b \\ cx &\leq \hat{z}_l \\ x &\in \mathbb{R}^n \end{aligned}$$

&

$$\begin{aligned} z_2^* &= \min -\hat{\pi}x \\ Ax &\geq b \\ cx &\leq \hat{z}_l \\ x &\in \mathbb{R}^n \end{aligned}$$

must have  
 $z_1^* > \hat{\pi}_0$ ,  
 $z_2^* > -\hat{\pi}_0 - 1$

## *Problem formulation: Maximum bound improvement*

If  $(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction,

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned}$$

&

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

must be  
infeasible

$$\begin{aligned} z_1^* &= \min \hat{\pi}x \\ Ax &\geq b \\ cx &\leq \hat{z}_l \\ x &\in \mathbb{R}^n \end{aligned}$$

&

$$\begin{aligned} z_2^* &= \min -\hat{\pi}x \\ Ax &\geq b \\ cx &\leq \hat{z}_l \\ x &\in \mathbb{R}^n \end{aligned}$$

must have  
 $z_1^* > \hat{\pi}_0$ ,  
 $z_2^* > -\hat{\pi}_0 - 1$

$$\begin{aligned} z_1^* &= \max pb - s_L \hat{z}_l \\ pA - s_L C &= -\hat{\pi} \\ p, s_L &\geq 0 \end{aligned}$$

&

$$\begin{aligned} z_2^* &= \max qb - s_R \hat{z}_l \\ qA - s_R C &= -\hat{\pi} \\ q, s_R &\geq 0 \end{aligned}$$

(as above)

## *Problem formulation: Maximum bound improvement*

If  $(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction,

$$\begin{array}{ll} z_1^* = \min \hat{\pi}x & z_2^* = \min -\hat{\pi}x \\ Ax \geq b & Ax \geq b \\ cx \leq \hat{z}_l & cx \leq \hat{z}_l \\ x \in \mathbb{R}^n & x \in \mathbb{R}^n \end{array} \quad \& \quad \begin{array}{l} \text{must have} \\ z_1^* > \hat{\pi}_0, \\ z_2^* > -\hat{\pi}_0 - 1 \end{array}$$

$$\begin{array}{ll} z_1^* = \max pb - s_L \hat{z}_l & z_2^* = \max qb - s_R \hat{z}_l \\ pA - s_L C = -\hat{\pi} & qA - s_R C = -\hat{\pi} \\ p, s_L \geq 0 & q, s_R \geq 0 \end{array} \quad \& \quad \begin{array}{l} \text{(as above)} \end{array}$$

$$\begin{array}{ll} pb - s_L \hat{z}_l > \hat{\pi}_0 & qb - s_R \hat{z}_l > -\hat{\pi}_0 - 1 \\ pA - s_L C = -\hat{\pi} & qA - s_R C = -\hat{\pi} \\ p, s_L \geq 0 & q, s_R \geq 0 \end{array} \quad \& \quad \begin{array}{l} \text{must be} \\ \text{feasible} \end{array}$$

## *Problem formulation: Maximum bound improvement*

If  $(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction,

$$\begin{aligned} z_1^* = \max \quad & pb - s_L \hat{z}_l \\ & pA - s_{LC} = -\hat{\pi} \quad \& \quad qA - s_{RC} = -\hat{\pi} \quad (\text{as above}) \\ & p, s_L \geq 0 \quad \quad \quad q, s_R \geq 0 \end{aligned}$$

$$\begin{aligned} pb - s_L \hat{z}_l &> \hat{\pi}_0 \\ pA - s_{LC} &= -\hat{\pi} \\ p, s_L &\geq 0 \end{aligned} \quad \& \quad \begin{aligned} qb - s_R \hat{z}_l &> -\hat{\pi}_0 - 1 \\ qA - s_{RC} &= -\hat{\pi} \\ q, s_R &\geq 0 \end{aligned} \quad \begin{array}{l} \text{must be} \\ \text{feasible} \end{array}$$

$$\begin{aligned} pb - s_L \hat{z}_l &> \hat{\pi}_0 \\ pA - s_{LC} &= -\hat{\pi} \\ qb - s_R \hat{z}_l &> -\hat{\pi}_0 - 1 \\ qA - s_{RC} &= -\hat{\pi} \\ q, s_R, p, s_L &\geq 0 \end{aligned}$$

## Problem formulation: Maximum bound improvement

If  $(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction,

$$\begin{array}{l} z_1^* = \max pb - s_L \hat{z}_l \\ pA - s_{LC} = -\hat{\pi} \\ p, s_L \geq 0 \end{array} \quad \& \quad \begin{array}{l} z_2^* = \max qb - s_R \hat{z}_l \\ qA - s_{RC} = -\hat{\pi} \\ q, s_R \geq 0 \end{array} \quad \text{(as above)}$$

$$\begin{array}{l} pb - s_L \hat{z}_l > \hat{\pi}_0 \\ pA - s_{LC} = -\hat{\pi} \\ p, s_L \geq 0 \end{array} \quad \& \quad \begin{array}{l} qb - s_R \hat{z}_l > -\hat{\pi}_0 - 1 \\ qA - s_{RC} = -\hat{\pi} \\ q, s_R \geq 0 \end{array} \quad \begin{array}{l} \text{must be} \\ \text{feasible} \end{array}$$

$$\begin{array}{l} pb - s_L \hat{z}_l > \hat{\pi}_0 \\ pA - s_{LC} = -\hat{\pi} \\ qb - s_R \hat{z}_l > -\hat{\pi}_0 - 1 \\ qA - s_{RC} = -\hat{\pi} \\ q, s_R, p, s_L \geq 0 \end{array} \quad \begin{array}{l} pb - s_L \hat{z}_l > \pi_0 \\ pA - s_{LC} = -\pi \\ qb - s_R \hat{z}_l > -\pi_0 - 1 \\ qA - s_{RC} = -\pi \\ q, s_R, p, s_L \geq 0 \\ (\pi, \pi_0) \in \mathbb{Z}^n \end{array}$$

## *Problem formulation: Maximum bound improvement*

If  $(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction,

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned}$$

&

$$\begin{aligned} Ax &\geq b \\ cx &\leq \hat{z}_l \\ \hat{\pi}x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

must be  
infeasible

$$\begin{aligned} pb - s_L \hat{z}_l &> \hat{\pi}_0 \\ pA - s_L C &= -\hat{\pi} \\ qb - s_R \hat{z}_l &> -\hat{\pi}_0 - 1 \\ qA - s_R C &= -\hat{\pi} \\ q, s_R, p, s_L &\geq 0 \end{aligned}$$

$$\begin{aligned} pb - s_L \hat{z}_l &> \pi_0 \\ pA - s_L C &= -\pi \\ qb - s_R \hat{z}_l &> -\pi_0 - 1 \\ qA - s_R C &= -\pi \\ q, s_R, p, s_L &\geq 0 \\ (\pi, \pi_0) &\in \mathbb{Z}^n \end{aligned}$$

$(\hat{\pi}, \hat{\pi}_0)$  is the required disjunction  $\iff$  it satisfies the last MIP.

## *Problem formulation: Maximum bound improvement*

$$\begin{aligned}pb - s_L \hat{z}_l &> \pi_0 \\pA - s_L C &= -\pi \\qb - s_R \hat{z}_l &> -\pi_0 - 1 \\qA - s_R C &= -\pi \\q, s_R, p, s_L &\geq 0 \\(\pi, \pi_0) &\in \mathbb{Z}^n\end{aligned}$$

- ▶  $2n + 2$  constraints,  $2m + n + 3$  variables.
- ▶ Tells us if we can increase the bound to  $\hat{z}_l$  by branching at the current node
- ▶ Does not tell us the maximum such value of  $\hat{z}_l$ .
- ▶ We do a binary search over a range of  $\hat{z}_l$  and solve the above MIP in each iteration

## *Problem formulation: Thin Directions*

Similar approach has been used before:

- ▶ Cut generating LPs (Balas)
- ▶ Separation of a given point from the *Split Closure* (Caprara, 2003, Balas 2007)
- ▶ **Finding thinnest directions?**

## Problem formulation: Thin Directions

Similar approach has been used before:

- ▶ Cut generating LPs (Balas)
- ▶ Separation of a given point from the *Split Closure* (Caprara, 2003, Balas 2007)
- ▶ Finding thinnest directions?

### Thin Directions

- ▶ Thickness of the LP relaxation polytope  $\mathcal{P}$  along a direction  $\hat{\pi}$  is:

$$\max \hat{\pi}y - \hat{\pi}x, \quad x, y \in \mathcal{P}$$

- ▶ If the width of  $\mathcal{P}$ ,  $w(\mathcal{P})$  is minimum such width over all  $\hat{\pi}$ :

$$w(\mathcal{Q}) = \min_{\pi} \max_{x, y \in \mathcal{P}} (\pi y - \pi x) \quad s.t. \quad \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

## *Problem formulation: Thin Directions*

$$w(Q) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi y - \pi x) \quad \text{s.t.} \quad \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

For a fixed  $(\hat{\pi}, \hat{\pi}_0)$ , the inner LP is:

$$\begin{aligned} \max \quad & \hat{\pi}y - \hat{\pi}x \\ \text{s.t.} \quad & Ax \geq b \\ & Ay \geq b \end{aligned}$$

Dual of the inner LP is:

$$\begin{aligned} \min \quad & -qb - pb \\ \text{s.t.} \quad & pA - \hat{\pi} = 0 \\ & qA + \hat{\pi} = 0 \\ & p, q \geq 0 \end{aligned}$$

Over all  $\pi$ :

$$\begin{aligned} \min \quad & -qb - pb \\ & pA - \pi = 0 \\ & qA + \pi = 0 \\ & p, q \geq 0 \end{aligned}$$

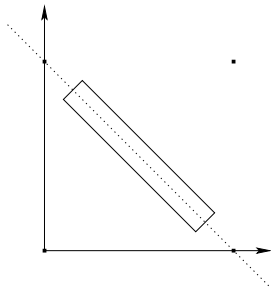
$$\pi \in \mathbb{Z}^n \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

Optional constraint:

$$\pi_0 < \pi x^* < \pi_0 + 1, \quad \pi_0 \in \mathbb{Z}$$

## More on Thin Directions

- ▶ Problem of finding the thinnest direction is  $NP$ -hard. Remains hard even if the given LP polytope is a simplex (Sebö, 1999)
- ▶ Branching on thinnest direction does not necessarily result in the smallest tree (Examples by Krishnamoorthy, 2008).
- ▶ Branching on the thinnest direction need not make the child nodes infeasible.
- ▶ If a general branching disjunction makes the children infeasible then  $w(\mathcal{P}) < 1$ . Converse is not true.



## Computational Experiments

- ▶ Objective: Study the effect of branching on *best* general hyperplanes on the size of branch-and-bound tree.
- ▶ We ignore time. Time spent on branching decisions  $\gg$  Time to solve the instance.
- ▶ Used ILOG CPLEX-10.2 MIP solver.
- ▶ Compared against strong branching (our own callback). Variable selected on the basis of maximum lower bound.
- ▶ Disabled Preprocessing, Heuristics.
- ▶ Cuts disabled in initial experiments.
- ▶ Provided best known upper bound to the solver.
- ▶ To perform experiments in reasonable time:
  - ▶ Imposed a limit of  $-M \leq \pi_i \leq M$  ( $M$  is a fixed parameter).
  - ▶ Imposed a constraint of  $|\sum_i \pi_i| \leq k$  ( $k$  is a fixed parameter).
  - ▶ Each MIP solved for finding a branching hyperplane had time limit  $t$  seconds ( $t$  is a fixed parameter).
  - ▶ Limit time on branching on a node to  $8T$  seconds
  - ▶ If more than 18 hours spent on branching, then switch to strong branching on variables.
  - ▶ Total time limit of 20 hours for each instance.

## Instances

30 Instances chosen from MIPLIB-3, MIPLIB-2003 and Mittlemann's collection.

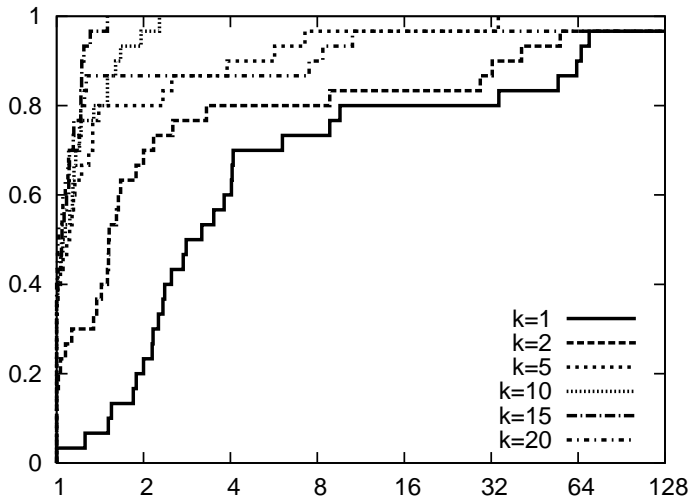
| Instance | Cons | Vars | Ints | Bins | Instance  | Cons | Vars  | Ints  | Bins  |
|----------|------|------|------|------|-----------|------|-------|-------|-------|
| 10teams  | 231  | 2025 | 1800 | 1800 | mod008    | 6    | 319   | 319   | 319   |
| aflow30a | 479  | 842  | 421  | 421  | neos6     | 1037 | 8768  | 8340  | 8340  |
| bell3a   | 123  | 133  | 71   | 39   | nug08     | 913  | 1632  | 1632  | 0     |
| blend2   | 274  | 353  | 264  | 231  | nw04      | 36   | 87482 | 87482 | 87482 |
| egout    | 98   | 141  | 55   | 55   | p0548     | 176  | 548   | 548   | 548   |
| fiber    | 363  | 1298 | 1254 | 1254 | pp08aCUTS | 246  | 240   | 64    | 64    |
| flugpl   | 18   | 18   | 11   | 0    | qnet1     | 503  | 1541  | 1417  | 1288  |
| gen      | 780  | 870  | 150  | 144  | qnet1_o   | 456  | 1541  | 1417  | 1288  |
| gesa2    | 1392 | 1224 | 408  | 240  | ran10x26  | 297  | 520   | 260   | 260   |
| gesa2_o  | 1248 | 1152 | 672  | 336  | ran12x21  | 286  | 504   | 502   | 502   |
| gt2      | 29   | 188  | 24   | 0    | ran13x13  | 196  | 338   | 169   | 169   |
| harp2    | 112  | 2993 | 2993 | 2993 | rout      | 291  | 556   | 315   | 300   |
| khh05250 | 101  | 1350 | 24   | 24   | stein45   | 331  | 45    | 45    | 45    |
| 1152lav  | 97   | 1989 | 1989 | 1989 | vpm1      | 234  | 378   | 168   | 168   |
| lseu     | 28   | 89   | 89   | 89   | vpm2      | 234  | 378   | 168   | 168   |

# First Experiment

| Instance  | $N_1$   | $N_5$  | $r_5$   | $N_{10}$ | $r_{10}$ | $N_{15}$ | $r_{15}$ | $N_{20}$ | $r_{20}$ |
|-----------|---------|--------|---------|----------|----------|----------|----------|----------|----------|
| 10teams   | 115     | 28     | 4.11    | 18       | 6.39     | 12       | 9.58     | 12       | 9.58     |
| aflow30a  | 36634   | 19485  | 1.88    | 20388    | 1.8      | 24112    | 1.52     | 20271    | 1.81     |
| bell3a    | 16387   | 8771   | 1.87    | 588      | 27.87    | 259      | 63.27    | 259      | 63.27    |
| blend2    | 304     | 231    | 1.32    | 188      | 1.62     | 165      | 1.84     | 209      | 1.45     |
| egout     | 2246    | 554    | 4.05    | 572      | 3.93     | 676      | 3.32     | 558      | 4.03     |
| fiber     | 18412   | 7612   | 2.42    | 3039     | 6.06     | 3358     | 5.48     | 3324     | 5.54     |
| flugpl    | 394     | 6      | 65.67   | 10       | 39.4     | 6        | 65.67    | 6        | 65.67    |
| gen       | 100     | 100    | 1       | 100      | 1        | 100      | 1        | 100      | 1        |
| gesa2     | 33526   | 21664  | 1.55    | 21849    | 1.53     | 21849    | 1.53     | 21778    | 1.54     |
| gesa2_o   | 98550   | 24435  | 4.03    | 24661    | 4        | 24661    | 4        | 24661    | 4        |
| gt2       | 340     | 10     | 34      | 12       | 28.33    | 10       | 34       | 12       | 28.33    |
| harp2     | 432010  | 174656 | 2.47    | 183306   | 2.36     | 174454   | 2.48     | 179130   | 2.41     |
| khh05250  | 738     | 594    | 1.24    | 588      | 1.26     | 614      | 1.2      | 618      | 1.19     |
| l152lav   | 60      | 32     | 1.88    | 28       | 2.14     | 34       | 1.76     | 30       | 2        |
| lseu      | 4058    | 226    | 17.96   | 78       | 52.03    | 58       | 69.97    | 58       | 69.97    |
| mod008    | 2840    | 296    | 9.59    | 102      | 27.84    | 68       | 41.76    | 52       | 54.62    |
| neos6     | 5989    | 2131   | 2.81    | 2131     | 2.81     | 2131     | 2.81     | 2131     | 2.81     |
| nug08     | 14      | 4      | 3.5     | 6        | 2.33     | 6        | 2.33     | 5        | 2.8      |
| nw04      | 30      | 16     | 1.88    | 12       | 2.5      | 12       | 2.5      | 12       | 2.5      |
| p0548     | 1050    | 466    | 2.25    | 566      | 1.86     | 565      | 1.86     | 565      | 1.86     |
| pp08aCUTS | 1301300 | 147271 | 8.84    | 166943   | 7.79     | 168905   | 7.7      | 231527   | 5.62     |
| qnet1     | 42      | 24     | 1.75    | 20       | 2.1      | 22       | 1.91     | 18       | 2.33     |
| qnet1_o   | 154     | 94     | 1.64    | 77       | 2        | 80       | 1.93     | 92       | 1.67     |
| ran10x26  | 68449   | 23309  | 2.94    | 24716    | 2.77     | 23704    | 2.89     | 21520    | 3.18     |
| ran12x21  | 494558  | 219967 | 2.25    | 208948   | 2.37     | 225980   | 2.19     | 212910   | 2.32     |
| ran13x13  | 124716  | 74699  | 1.67    | 57825    | 2.16     | 66008    | 1.89     | 58789    | 2.12     |
| rout      | 219322  | 65201  | 3.36    | 61806    | 3.55     | 61226    | 3.58     | 57673    | 3.8      |
| stein45   | 31086   | 21238  | 1.46    | 20594    | 1.51     | 20601    | 1.51     | 20601    | 1.51     |
| vpm1      | 263111  | 145    | 1814.56 | 32       | 8222.22  | 20       | 13155.55 | 5929     | 44.38    |
| vpm2      | 273994  | 77504  | 3.54    | 67014    | 4.09     | 69515    | 3.94     | 73687    | 3.72     |

Number of nodes ( $N_k$ ) in tree and the ratio  $r_k = \frac{N_1}{N_k}$  for selected instances when  $t = 1000s$ ,  $M = 1$ ,  $k$  is varied.

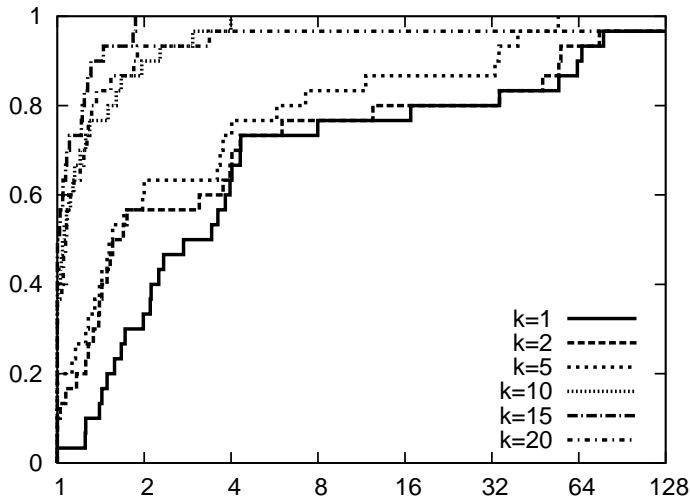
## Using performance profiles\*



Performance profile for number of subproblems when  $t$  is fixed and  $k$  is varied.  $M = 1$

\*Dolan, 2002

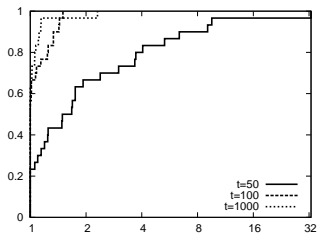
## Using performance profiles\*



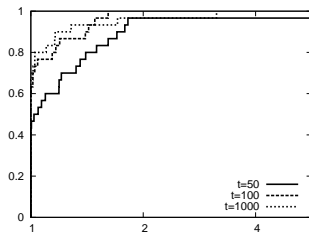
Performance profile for number of subproblems when  $t$  is fixed and  $k$  is varied.  $M = 1$

\*Dolan, 2002

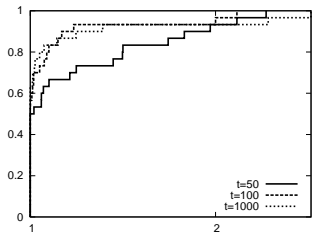
## Dependence on $t$



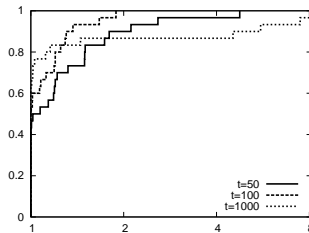
(a)  $k = 5$



(b)  $k = 10$



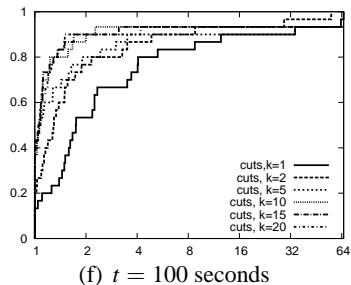
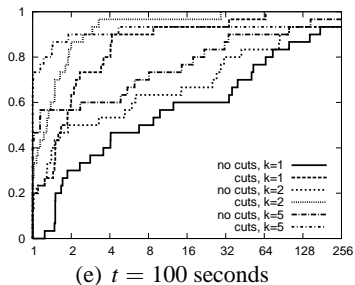
(c)  $k = 15$



(d)  $k = 20$

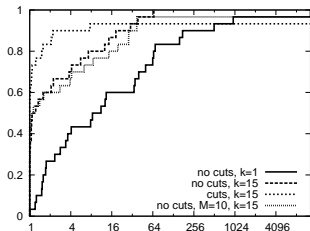
Performance profile for number of subproblems when  $k$  is fixed and  $t$  (in seconds) is varied.

## *In presence of cuts*

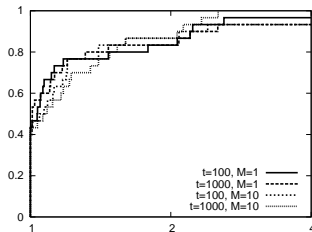


*Figure:* Performance profile for number of subproblems when cuts are added to the original problem.  $t$  is fixed and  $k$  is varied.

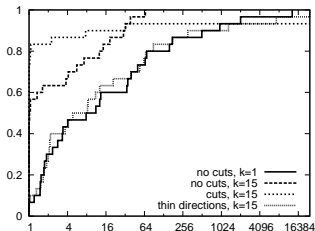
## Having higher coefficients and thin directions



$M$  is increased to 10,  $t = 100s$



Effects of changing  $t$  when  $M$  is increased,  
 $t = 100$  seconds



Comparison of effect of branching on *thin* directions against other criteria,  $t = 100$  seconds

## *Concluding remarks*

- ▶ The problem of finding a general disjunction that maximizes the lower bound is formulated
- ▶ The problem is shown to be  $\mathcal{NP}$ -hard in the general case and also for several specific restrictions
- ▶ A MIP formulation that can be solved to find such disjunctions was shown
- ▶ Computational experiments show that these disjunctions can reduce the size of branch-and-bound tree by a significant amount.
- ▶ The time taken to solve these MIPs is prohibitively large
- ▶ We are currently working on techniques to speed up this approach and develop heuristics
  
- ▶ A.M. and T.K. Ralphs, *Experiments with Branching using General Disjunctions*, Proceedings of the Eleventh INFORMS Computing Society Meeting, 2009, To Appear
- ▶ A.M. and T.K. Ralphs, *On the complexity of selecting branching disjunctions for integer programming*, Working Paper