On the Complexity of Selecting Branching Disjunctions in Integer Programming

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Abstract

Branching is an important component of branch-and-bound algorithms for solving mixed integer linear programs. We consider the problem of selecting, at each iteration of the branch-and-bound algorithm, a general branching disjunction of the form "$\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$", where π, π_0 are integral. We show that the problem of selecting an optimal such disjunction, according to specific criteria described herein, is $\mathcal{NP}$-hard. We further show that the problem remains $\mathcal{NP}$-hard even for binary programs or when considering certain restricted classes of disjunctions. We observe that the problem of deciding whether a given inequality is a split inequality can be reduced to one of the above problems, which leads to a proof that this problem is $\mathcal{NP}$-complete.

1 Introduction

In this paper, we study the computational complexity of the problem of selecting "optimal" branching disjunctions in the branch-and-bound algorithm for solving mixed integer linear programs (MILPs). The motivation for studying the complexity of these problems is that such models may be useful in guiding branching decisions in a practical context. In particular, these problems can, in principle, be solved at each node of the branch-and-bound tree to select the "optimal" disjunction for partitioning the feasible region of that node. In an earlier paper [Mahajan and Ralphs, 2009], we showed by means of several experiments that the number of nodes in the branch-and-bound tree can be reduced significantly by employing such selection procedures. However, the time required to solve these problems, when formulated as straightforward optimization problems, using a generic solver, is prohibitively large. In this paper, we show that these problems in fact lie in the complexity class $\mathcal{NP}$-hard, even for binary MILPs and even when certain restrictions are imposed on the structure of the disjunctions. It is well-known that the disjunctions used for branching may also be used to generate valid inequalities, called *split inequalities*. We show that the problem of deciding whether a given inequality is an "elementary" split inequality can be reduced to a problem related to determining an optimal branching disjunction. This immediately leads to a proof that the problem of deciding whether a given inequality is an elementary split inequality is $\mathcal{NP}-$complete.

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In order to define the problem of selecting an optimal disjunction, we first need to briefly describe the branch-and-bound algorithm. Consider the mathematical program

$$\begin{aligned} \min\ & cx \\ \text{s.t.}\ Ax\ & \geq b \\ x\ & \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{aligned} \tag{P}$$

where $b \in \mathbb{Q}^m, c \in \mathbb{Q}^n, A \in \mathbb{Q}^{m \times n}$ are inputs and where the variables with indices $1, 2, \ldots, d$ are constrained to be integer-valued. The linear programming (LP) relaxation of (P), obtained by relaxing the integrality requirement, can be written as

$$\min_{x \in \mathcal{P}} cx, \tag{1}$$

where $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \geq b\}$. An LP-based branch-and-bound algorithm for solving (P) starts by solving the LP relaxation in order to obtain a lower bound on the objective function value (taken to be ∞ if $\mathcal{P}$ is empty). If the solution, x^* to the LP relaxation is a member of $\mathbb{Z}^d \times \mathbb{R}^{n-d}$, then x^* is an optimal solution for (P) as well and we are done. Otherwise, we determine a disjunction (usually binary) that is satisfied by all solutions to (P) but not satisfied by x^*. Such a disjunction, referred to henceforth as a *valid branching disjunction* or simply a *valid disjunction*, divides the feasible region into (usually disjoint) subsets. The algorithm can then be applied recursively to the subproblems associated with these subsets until exhaustion.

A feasible solution to (P) provides an upper bound on the optimal objective value. Such solutions can be found by heuristics or after solving the LP relaxation associated with a given subproblem. If the optimal objective function value of an LP relaxation associated with a subproblem is greater than the upper bound, then the subproblem may be discarded. For a more complete and formal description, see [Nemhauser and Wolsey, 1988]. Note that the term *subproblem* is associated with a restriction of the original instance (P) resulting from the imposition of one or more branching disjunctions. These subproblems should not be confused with the associated problem of selecting a branching disjunction, which is formulated in the following sections.

In order to define formally the problem of selecting an optimal disjunction, we must first define the set of disjunctions to be considered. A branching disjunction of the form "$x_i \leq k \vee x_i \geq k+1$" for any $1 \leq i \leq d, k \in \mathbb{Z}$ is always valid for (P). Such a disjunction will be called a *variable disjunction*. Most implementations of branch-and-bound use only variable disjunctions for branching. More generally, however, any $\pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi_0 \in \mathbb{Z}$ yields a disjunction "$\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$". Such a disjunction, referred to henceforth as a *general disjunction* and denoted by the ordered pair (π, π_0), is also always valid. Since the set of general disjunctions includes all variable disjunctions, selecting a disjunction from this set should, in principle, be more effective. In this paper, we study the complexity of the problem of selecting the optimal general disjunction over this set. Before doing that, it is necessary to describe the criteria for selecting a branching disjunction.

In its simplest form, the efficiency of the branch-and-bound procedure depends mainly on the number of subproblems generated. The goal of selecting a branching disjunction is then to minimize the total number of subproblems to be solved. Liberatore [2000] showed in the context of Satisfiability Problems (SATs) that the problem of finding optimal variable disjunction (according to the criteria of minimizing the overall size of the search tree) is $\mathcal{NP}$-hard. Since SATs are reducible, in polynomial time, to MILPs, a similar result may be expected for the case of MILPs. In light of this, the problem of selecting an optimal general branching disjunction appears to be difficult. The approach taken by most solution procedures, and the one we shall take here, is then to evaluate candidate branching disjunctions by assessing their effect using more myopic criteria.

The primary criterion for selecting variable disjunctions studied in the literature so far is that of bound improvement. Both Linderoth and Savelsbergh [1999] and Achterberg et al. [2005] showed empirically that selecting a variable disjunction that leads to the maximum increase in the bound of the subproblems can reduce the number of subproblems required to solve the problem. Some other heuristic procedures to select variable disjunctions have been studied by, among others, Patel and Chinneck [2007] and Gilpin and Sandholm [2007].

On the other hand, the criteria for selecting general disjunctions have primarily been limited to "integer width" of the feasible region associated with the LP relaxation. In their survey, Aardal and Eisenbrand [2004] discussed the fact that when the dimension is fixed, polynomial time algorithms for solving integer programs can be obtained by branching on general disjunctions obtained by determining the so-called *thin directions* of the feasible region, i.e., disjunctions along which the integer width of the feasible region is small. These polynomial time algorithms are derived from the seminal work of Lenstra [H.W. Lenstra, 1983] and its extensions. It has also been shown, for instance by Krishnamoorthy and Pataki [2006], that certain specific problems can be solved "easily" if one branches on particular general disjunctions. Some heuristics that may enhance the computational efficiency of branch-and-cut algorithms by branching on general disjunctions have also been proposed recently. Fischetti and Lodi [2003] proposed a primal heuristic in which the search space is generated from a general disjunction. Owen and Mehrotra [2001], Karamanov and Cornuéjols [2007] and Cornuéjols et al. [2008] proposed heuristics to identify a set of general disjunctions as candidates for branching. They select, from this set, the one that improves the associated bound by the maximum.

The remainder of the paper is organized as follows. We describe the problem of finding the optimal disjunction with respect to both the criteria of maximal bound improvement and minimum integer width in Section 2 and also provide MILP formulations that could be used to obtain an optimal disjunction based on these criteria. The computational complexity of these problems is discussed in Section 4. The results obtained there are then used to derive a result for the problem of deciding whether a given inequality is an elementary split inequality in Section 4. Finally, in Section 5, we present conclusions.

2 Selecting Branching Disjunctions

Before addressing the complexity of the problem of selecting a branching disjunction, we first describe the problem of selecting a general branching disjunction to maximize the bound improvement and show how to construct a sequence of MILPs to solve it. We then show, using a similar approach, that the problem of selecting the thinnest direction can be also formulated as an MILP.

2.1 Maximizing Bound Improvement

The problem of selecting a disjunction to maximize bound improvement can be stated as follows. Consider the integer program (P) and assume that the associated polyhedron $\mathcal{P}$ is nonempty. Let $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^d \times \{0\}^{n-d} \times \mathbb{Z}^1$ be a given branching disjunction. Then the LPs associated with the subproblems created after branching are

$$
\begin{array}{lll}
z_L^* = \min cx & & z_R^* = \min cx \\
\text{s.t.} \quad Ax \geq b & \text{and} & \text{s.t.} \quad Ax \geq b \\
\qquad \hat{\pi}x \leq \hat{\pi}_0 & & \qquad \hat{\pi}x \geq \hat{\pi}_0 + 1.
\end{array}
\tag{2}
$$

Problem 1 (Greatest lower bound from a general branching disjunction)**.** *Given a mathematical program of the form (P) and $z_l \in \mathbb{R}$, find $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^d \times \{0\}^{n-d} \times \mathbb{Z}^1$ such that $\min\{z_L^*, z_R^*\}$ is maximized, where z_L^*, z_R^* are as defined in (2).*

Problem 1 is an optimization problem and the associated decision problem is as follows.

Problem 2 (Lower bound from a general branching disjunction)**.** *Given a mathematical program of the form (P) and $z_l \in \mathbb{R}$, does there exist $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^d \times \{0\}^{n-d} \times \mathbb{Z}^1$ such that the LP relaxation associated with each subproblem (2) created after branching on $(\hat{\pi}, \hat{\pi}_0)$ has an optimal objective value of at least z_l, i.e., $\min\{z_L^*, z_R^*\} \geq z_l$?*

Here, we describe a procedure to solve Problem 2 and then show in Section 3 that the problem is $\mathcal{NP}$-complete. We first consider a special case of this problem when the feasible region of the LPs associated with each subproblem is empty (i.e., $z_l = \infty$) and then extend the results to the general case. Consider the following problem.

Problem 3 (Disjunctive infeasibility)**.** *Given a mathematical program of the form (P), does there exist $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^d \times \{0\}^{n-d} \times \mathbb{Z}^1$ such that the feasible region of the LP relaxations associated with each subproblem (2) created after branching on $(\hat{\pi}, \hat{\pi}_0)$ is empty, i.e., $\min\{z_L^*, z_R^*\} = \infty$?*

The solution to Problem 3 does not depend upon the cost vector c because it is only desired to prove that the problem (P) is infeasible. The problem of finding a desired disjunction $(\hat{\pi}, \hat{\pi}_0)$ can be formulated as follows. Assume again that $\mathcal{P}$ is nonempty. Suppose $(\hat{\pi}, \hat{\pi}_0)$ is chosen such that both LPs (2) become infeasible. Then consider the following problems:

$$
\begin{array}{lll}
\zeta_L^* = \min \hat{\pi}x & \text{and} & \zeta_R^* = \min -\hat{\pi}x \\
\text{s.t.} \quad Ax \geq b & & \text{s.t.} \quad Ax \geq b.
\end{array}
\tag{3}
$$

The dual of each of the above two programs (3) can be written, respectively, as:

$$
\begin{array}{lll}
\zeta_L^* = \max pb & & \zeta_R^* = \max qb \\
\text{s.t.} \quad pA = \hat{\pi} & \text{and} & \text{s.t.} \quad qA = -\hat{\pi} \\
\qquad p \geq 0 & & \qquad q \geq 0.
\end{array}
\tag{4}
$$

The programs (2) are infeasible if and only if $\zeta_L^* > \hat{\pi}_0$ and $\zeta_R^* > -(\hat{\pi}_0 + 1)$. By using this condition and combining the above two dual formulations, one can get the desired formulation for giving an answer to Problem 3. More precisely, the LPs in (2) are infeasible if and only if the system

$$
\begin{aligned}
pA - \pi &= 0 \\
qA + \pi &= 0 \\
pb - \pi_0 &> 0 \\
qb + \pi_0 &> -1 \\
p &\geq 0 \\
q &\geq 0 \\
(\pi, \pi_0) &\in \mathbb{Z}^d \times \{0\}^{n-d} \times \mathbb{Z}^1,
\end{aligned}
\tag{5}
$$

has a feasible solution with $\pi = \hat{\pi}, \pi_0 = \hat{\pi}_0$.

Once we have a formulation that may be solved in order to answer Problem 3, we can extend it in the usual way to address Problem 2 as well. More details about this procedure are described in Mahajan and Ralphs [2009]. Problem 3 is equivalent to that of finding $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^d \times \{0\}^{n-d} \times \mathbb{Z}^1$ such that $\mathcal{P} \subseteq \{x \in \mathbb{R}^n \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$. If such a $(\hat{\pi}, \hat{\pi}_0)$ exists, then the "width" $\mathcal{P}$ is less than one. In the next section, we study the problem of minimizing the integer width of $\mathcal{P}$ and show that the general framework described above can be extended to this problem as well.

2.2 Minimizing Integer Width

Assuming that $\mathcal{P}$ is full dimensional, the *width* of $\mathcal{P}$ in direction π is $\max_{x,y \in \mathcal{P}}(\pi x - \pi y)$, while the *integer width* of $\mathcal{P}$ is

$$w(\mathcal{P}) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi x - \pi y), \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq 0.$$

Then, a vector π that is obtained from the above optimization problem, along with a scalar $\pi_0 = \lfloor \pi x^* \rfloor$, where x^* is the optimal solution of the LP relaxation (1), can be used to determine a disjunction for branching. Sebő [1999] showed that the problem of determining whether $w(\mathcal{P}) \leq 1$ is $\mathcal{NP}$-complete, even when $\mathcal{P}$ is a simplex. It is also known, from a result of Banaszczyk et al. [1999], that if $\mathcal{P} \cap \mathbb{Z}^{n-d}$ is empty, then $w(\mathcal{P}) \leq Cn^{\frac{3}{2}}$, where C is a constant. Derpich and Vera [2006] approximate the direction of the minimum integer width in order to assign priorities for branching on variables and use this to select variable disjunctions.

The width of $\mathcal{P}$ in a given direction $\hat{\pi}$ can be obtained by solving the LP

$$\begin{aligned} \max\ & \hat{\pi}x - \hat{\pi}y \\ \text{s.t.}\ & Ax \geq b \\ & Ay \geq b. \end{aligned} \tag{6}$$

The dual associated with the LP (6) is

$$\begin{aligned} \min\ & -qb - pb \\ \text{s.t.}\ & pA = \hat{\pi} \\ & qA = -\hat{\pi} \\ & p, q \geq 0. \end{aligned} \tag{7}$$

Therefore, the problem of finding $w(\mathcal{P})$ can be written as

$$\begin{aligned} \min\ & -qb - pb \\ \text{s.t.}\ & pA - \pi = 0 \\ & qA + \pi = 0 \\ & \pi \neq 0 \\ & \pi \in \mathbb{Z}^d \times \{0\}^{n-d} \\ & p, q \geq 0. \end{aligned} \tag{8}$$

Since the disjunction (π, π_0) is the same as the disjunction $(-\pi, -\pi_0 - 1)$, the condition $\pi \neq 0$ can be replaced by the inequality $\sum_{i=1}^{n} \pi_i \geq 1$. Problem (8) can now be solved as an MILP.

Note that if there exists a $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$ that satisfies the formulation (5), then $w(\mathcal{P}) < 1$. However, the converse is not true. To see this, consider as an example $\mathcal{P} = \{x \in \mathbb{R}^2_+ \mid 3 \leq 4x_1 + 4x_2 \leq 5\}$ and $d = 2$. Then, even though $w(\mathcal{P}) \leq \frac{1}{2} < 1$, (P) is still feasible. Comparing formulations (5) and (8), one can see that (5) is more constrained than (8). As a result, there may be some benefit to using solutions to the formulation (5) to generate branching disjunctions over those of (8). A feasible solution to formulation (5) guarantees that the LP relaxations associated with both subproblems created after branching are infeasible and therefore gives a short proof of infeasibility, provided that such a short proof exists. Branching along a direction of minimum width does not guarantee this. As an example, consider an MILP with feasible region $\{x \in \mathbb{Z}^2_+ \mid 7 \leq 8x_1 + 8x_2 \leq 9, -3 \leq 4x_1 - 4x_2 \leq 3\}$. Branching on the disjunction $x_1 \leq 0 \vee x_1 \geq 1$ immediately makes LP relaxation of each subproblem infeasible while branching along a direction of minimum width ($w(\mathcal{P}) = 0.25$), $x_1 + x_2 \geq 2 \vee x_1 + x_2 \leq 1$ results in two subproblems out of which one still has a feasible LP relaxation and needs further processing. Krishnamoorthy [2008] showed that, in general, branching along a direction of minimum width need not result in a small branch-and-bound tree, even in higher dimensions.

Even though there are some similarities in the formulations (5) and (8), it is not easy to reduce the problem of finding $w(\mathcal{P})$ to Problem 3. Therefore, we use a different approach to address the complexity of the latter.

3 Complexity Analysis

For the case when (π, π_0) is restricted to variable disjunctions only, Problem 3 can be solved in time polynomial in the size of the input by solving the two LPs associated with each of the n possible variable disjunctions. The following results show that the problem becomes difficult in the case of general disjunctions. We first show that Problem 3 is $\mathcal{NP}$-complete. We then show that the problem remains $\mathcal{NP}$-complete even when several common restrictions are introduced.

Lemma 3.1. *If $(\hat{\pi}, \hat{\pi}_0, \hat{p}, \hat{q})$ is a feasible solution to (5), then $\hat{\pi}_0 < \hat{p}b \leq -\hat{q}b < \hat{\pi}_0 + 1$.*

Proof. The first and last inequalities come directly from the formulation (5). Let $\zeta^*_L = \min_x\{\hat{\pi}x \mid Ax \geq b\}, \zeta^*_R = \max_x\{\hat{\pi}x \mid Ax \geq b\}$. Then $\zeta^*_L \leq \zeta^*_R$. Also, $p = \hat{p}$ and $q = \hat{q}$ are feasible solutions to the dual programs (4). By using weak duality on the associated LPs (3), we get that $\zeta^*_L \geq \hat{p}b$ and $\zeta^*_R \leq -\hat{q}b$. Thus, $\hat{\pi}_0 < \hat{p}b \leq \zeta^*_L \leq \zeta^*_R \leq -\hat{q}b < \hat{\pi}_0 + 1$. □

We assume for the remainder of the section that the mathematical program (P) is a pure integer program, i.e., that $n = d$. This assumption is made for notational convenience only—the results are also valid for the case when $d < n$.

We first show that the Problem 3 is in the complexity class $\mathcal{NP}$. If the matrices A, b have integer entries only, then we claim that constraints $p < \mathbf{1}$, where $\mathbf{1}$ is the vector of all ones, may be added to (5) without any loss of generality. In order to see this, suppose the formulation (5) has a feasible solution with $p = \hat{p}, q = \hat{q}, \pi = \hat{\pi}, \pi_0 = \hat{\pi}_0$. Further suppose that $\hat{p}_i \geq 1$ for some $i, 1 \leq i \leq m$. Then $p = \hat{p} - e_i, q = \hat{q} + e_i, \pi = \hat{\pi} - a_i, \pi_0 = \hat{\pi}_0 - b_i$ is also a feasible solution. Here, e_i is the i^{th} unit vector and a_i the i^{th} row of the matrix A. This process can be applied repeatedly until p is component-wise less than 1. If we assume that $p < \mathbf{1}$, then $|pb| \in [0, \sum_{i=1}^m |b_i|)$. Also, using (5), $|\pi_j| \in [0, \sum_{i=1}^m |a_{ij}|), j = 1, \dots, n$. Using Lemma 3.1, this implies $|\pi_0| \leq |pb| \leq \sum_{i=1}^m |b_i|$. So, if the system (5) is feasible, then a feasible solution may be expressed in size that is polynomial in the size of the input. Also, given a $(\hat{\pi}, \hat{\pi}_0)$, one can determine whether a disjunction on $(\hat{\pi}, \hat{\pi}_0)$ will make the LPs (2) infeasible in time that is polynomial in the size of the input by solving the two linear programs. This shows that Problem 3 lies in the complexity class $\mathcal{NP}$.

Before further addressing the complexity of Problem 3, we consider the same problem applied to a system of linear Diophantine equations in place of the system of form (P). Suppose we are given a system of linear Diophantine equations of the form,

$$\begin{aligned} Ax &= b \\ x &\in \mathbb{Z}^n. \end{aligned} \tag{9}$$

Such equations can be solved in time polynomial in the size of the input [Nemhauser and Wolsey, 1988, pg. 191]. A branching disjunction $(\hat{\pi}, \hat{\pi}_0)$ that can make the associated LP relaxations of (9) infeasible can be shown, by using the approach above, to satisfy (along with a suitable $\hat{p}, \hat{q}$) the system

$$\begin{aligned} pA &= \pi, \\ -qA &= \pi, \\ pb &> \pi_0, \\ -qb &< 1 + \pi_0, \text{ and} \\ (\pi, \pi_0) &\in \mathbb{Z}^{n+1}. \end{aligned} \tag{10}$$

We claim that the system (10) can be solved in time polynomial in the size of the input. The system of Diophantine equations (9) is infeasible if and only if there exists a λ such that $\lambda A \in \mathbb{Z}^m$ and $\lambda b \notin \mathbb{Z}$ [Nemhauser and Wolsey, 1988, pg. 191]. Further, if (9) is infeasible, then such a λ can be found in polynomial time. In such a case, $p = -q = \lambda, \pi = \lambda A, \pi_0 = \lfloor \lambda b \rfloor$ is a feasible solution to (10). Conversely, suppose that (9) has a feasible solution x^0. Such a feasible solution can be found in polynomial time. Then for any $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$, $\pi x^0 = pAx^0 = pb$ and $\pi x^0 = -qAx^0 = -qb$. Since $\pi x^0 \in \mathbb{Z}$, there is no $\pi_0 \in \mathbb{Z}$ such that $\pi_0 > \pi x^0$ and $\pi_0 < \pi x^0 + 1$. Thus, in this case, the existence of the solution x^0 is sufficient to show that (10) is infeasible. So a feasible solution for the system (10) can be found or it can be shown that no such solution exists in time polynomial in the size of the input.

Now consider the problem (P) again (recall the assumption that $n = d$). We just have shown that the problem of finding $\lambda \in \mathbb{R}^m$ such that $\lambda A \in \mathbb{Z}^n, \lambda b \notin \mathbb{Z}$ is easy. Existence of such a λ is a necessary condition for the feasibility of a given program of the form (5). To see this, suppose $p = \hat{p}, \pi = \hat{\pi}, \pi_0 = \hat{\pi}_0$ are feasible for (5) and substitute $\lambda = \hat{p}$. Then $\lambda A = \hat{\pi} \in \mathbb{Z}^n$, but $\hat{\pi}_0 < \lambda b < \hat{\pi}_0 + 1$. Existence of such a λ is not, however, sufficient for feasibility of (5). For instance, consider the set: $\mathcal{P} \cap \mathbb{Z}^2 = \{x \in \mathbb{Z}^2 | 3x_1 + 6x_2 \geq 2\}$ and $\lambda = \frac{1}{3}$. Clearly $\lambda A \in \mathbb{Z}^2, \lambda b \notin \mathbb{Z}$. Still, $\mathcal{P} \cap \mathbb{Z}^n$ has at least one feasible point $(1, 0)$. This provides a hint that Problem 3 may not be easy.

We now show that Problem 3 is $\mathcal{NP}$-complete by reducing the well-known number partitioning problem to Problem 3. The Number Partitioning Problem $PARTITION$ is defined as follows

Problem 4 ($PARTITION$, [Garey and Johnson, 1979]). *Given a finite set S and a size $a^i \in \mathbb{Z}_+$ for each $i \in S$. Is there a subset $K \subseteq S$ such that $\sum_{i \in K} a^i = \sum_{i \in S \setminus K} a^i$?*

Proposition 3.2. *Problem 3 is $\mathcal{NP}$-complete.*

Proof. The proof is a modification of the approach used by Sebő [1999] for the problem of finding integer width. Consider the Problem 4 above, which is known to be $\mathcal{NP}$-complete. Let $n \in \mathbb{N}$, $S = \{1, 2, \ldots, (n-1)\}, a^i \in \mathbb{Z}_+, i \in S$ be inputs for Problem 4. Let $s = \frac{1}{2}\sum_{i \in S} a^i$. An instance of Problem 4 can be answered "yes" if and only if there exists a set $K \subseteq \{1, 2, \ldots, n-1\}$ such that $\sum_{i \in K} a^i = s$. Since multiplying each a^i by 4 results in a problem equivalent to Problem 4, it is assumed, without loss of generality, that $s \in \mathbb{Z}_+, s \geq 2$. Problem 4 can be reduced to Problem 3 as follows. Consider the simplex $\mathcal{P}_s$ of points $v^i, i = 1 \ldots n+1$ in n dimensions, with the coordinates of v^i defined as

$$v^i_j = \begin{cases} \frac{1}{2n} & \text{if } j \neq i, i = 1, 2, \ldots, n, \\ \frac{1}{2n} + \frac{1}{2} & \text{if } j = i, i = 1, 2, \ldots, n, \\ a^j & \text{if } i = n+1, j = 1, 2, \ldots, n-1, \\ -\frac{1}{2}\sum_{k=1}^{n-1} a^k + \frac{1}{2} & \text{if } i = n+1, j = n. \end{cases}$$

So, $v^1 = (\frac{1}{2n} + \frac{1}{2}, \frac{1}{2n}, \frac{1}{2n}, \ldots, \frac{1}{2n}), v^2 = (\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}, \frac{1}{2n}, \ldots, \frac{1}{2n}), \ldots, v^n = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2n} + \frac{1}{2}), v^{n+1} = (a^1, a^2, \ldots, a^{n-1}, -s + \frac{1}{2})$. We will show that the desired subset K exists if and only if there exists $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$, such that $\mathcal{P}_s \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$.

Suppose the desired subset K exists, i.e., K is a set such that $\sum_{i \in K} a^i = s$. Let $\hat{\pi}_i = 1, i \in K$, $\hat{\pi}_n = 1$, $\hat{\pi}_i = 0, i \notin K \cup \{n\}$, $\hat{\pi}_0 = 0$. Then, $0 < \hat{\pi}v^i < 1, i = 1, 2, \ldots, n$. Also, $v^{n+1}\hat{\pi} = \frac{1}{2}$. Since all vertices of $\mathcal{P}_s$ satisfy the condition $\hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1$, $\mathcal{P}_s \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$. $(\hat{\pi}, \hat{\pi}_0)$ is then the required disjunction.

Conversely, suppose there exists some $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$ such that $\mathcal{P}_s \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$. Then, $\hat{\pi}_0 < \hat{\pi}v^i < \hat{\pi}_0 + 1, i = 1, 2, \ldots, (n+1)$ and $|\hat{\pi}(v^i - v^k)| < 1, i = 1, 2, \ldots, n, k = 1, 2, \ldots, n$. Substituting the coordinates of v^i and v^k, one gets that $|\frac{\hat{\pi}_i - \hat{\pi}_k}{2}| < 1$. Since $\hat{\pi}_i, \hat{\pi}_k \in \mathbb{Z}$, this means that $|\hat{\pi}_i - \hat{\pi}_k| \leq 1$ for each pair $(i, k) \in \{1, 2, \ldots, n\}^2$. So, $\hat{\pi}_i \in \{t, t+1\}, i = 1, 2, \ldots, n$ for some $t \in \mathbb{Z}$. Since disjunction $(\hat{\pi}, \hat{\pi}_0)$ is equivalent to disjunction $(-\hat{\pi}, -\hat{\pi}_0 - 1)$, it can be assumed without loss of generality that $t \geq 0$. Let $K = \{i \mid \hat{\pi}_i = t+1\}$. Substituting the coordinates of v^1 and $\hat{\pi}$ into the inequalities $\hat{\pi}_0 < v^1\hat{\pi} < \hat{\pi}_0 + 1$, one gets

$$\begin{aligned} \hat{\pi}_0 < & \sum_{i=1}^{n} v^1_i \hat{\pi}_i < \hat{\pi}_0 + 1 \\ \Rightarrow \hat{\pi}_0 < & \sum_{i=1}^{n} \frac{\hat{\pi}_i}{2n} + \frac{\hat{\pi}_1}{2} < \hat{\pi}_0 + 1 \\ \Rightarrow \hat{\pi}_0 < & \frac{t}{2} + \sum_{i \in K} \frac{1}{2n} + \frac{\hat{\pi}_1}{2} < \hat{\pi}_0 + 1. \end{aligned}$$

Since $\frac{\hat{\pi}_1}{2} \in \{\frac{t}{2}, \frac{t+1}{2}\}$, the only integer value of $\hat{\pi}_0$ that satisfies the above condition is $\hat{\pi}_0 = t$. Thus $\hat{\pi} \in \{t, t+1\}^n, \hat{\pi}_0 = t$. Also, $K = \phi$ would mean that $\hat{\pi}_0 < t < \hat{\pi}_0 + 1$. This is not possible for any integers $t, \hat{\pi}_0$. Hence K is not empty. The condition $\hat{\pi}_0 < \hat{\pi} v^{n+1} < \hat{\pi}_0$ implies:

$$\begin{aligned}
\hat{\pi}_0 < & \sum_{i=1}^{n} v_i^{n+1} \hat{\pi}_i < \hat{\pi}_0 + 1 \\
\Rightarrow \hat{\pi}_0 < & \sum_{i=1}^{n-1} \hat{\pi}_i a^i - \hat{\pi}_n s + \frac{\hat{\pi}_n}{2} < \hat{\pi}_0 + 1 \\
\Rightarrow t \ < & \ t \sum_{i=1}^{n-1} a^i + \sum_{i \in K} a^i - \hat{\pi}_n s + \frac{\hat{\pi}_n}{2} < t + 1 \\
\Rightarrow t \ < & \ 2ts + \sum_{i \in K} a^i - \hat{\pi}_n s + \frac{\hat{\pi}_n}{2} < t + 1.
\end{aligned}$$

Now there are two cases. Suppose $\hat{\pi}_n = t$. Then the above condition implies that $t < ts + \sum_{i \in K} a^i + \frac{t}{2} < t + 1$. This is not possible because $s \geq 2$ and $K \neq \phi$. Thus, $\hat{\pi}_n$ must equal $t + 1$. In this case, the above condition becomes:

$$\begin{aligned}
t < 2ts + \sum_{i \in K} a^i - ts - s + \frac{t+1}{2} < t + 1 \\
\Rightarrow t < \quad (t-1)s + \sum_{i \in K} a^i + \frac{t+1}{2} < t + 1.
\end{aligned}$$

Since $s \geq 2$ and $K \neq \phi$, the only value that t may assume is $t = 0$. That means $0 < \sum_{i \in K} a^i - s + \frac{1}{2} < 1$. Thus $\sum_{i \in K} a^i = s$ and K is the required subset for the Problem 4.

Thus, given a simplex $\mathcal{P}_s$, the problem of finding $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$ is $\mathcal{NP}$-complete. Since $\mathcal{P}_s$ is a simplex, its description can be transformed into form (P) in time polynomial in the size of the description of $\mathcal{P}_s$. This completes the required proof. □

Even though the above proof did not assume any restrictions on values of $(\hat{\pi}, \hat{\pi}_0)$, the reduction from Problem 4 imposed the conditions $\hat{\pi} \in \{0,1\}^n$. This shows that several restrictions of Problem 3 are also $\mathcal{NP}$-complete. Some of these are listed below

Proposition 3.3. *The following restrictions of Problem 3 are $\mathcal{NP}$-complete.*

1. *Given a mathematical program of the form (P), does there exist $(\hat{\pi}, \hat{\pi}_0) \in \{0,1\}^{n+1}$ such that both the subproblems (2) created after branching on $(\hat{\pi}, \hat{\pi}_0)$ are infeasible.*
2. *Given a mathematical program of the form (P), does there exist $\hat{\pi} \in \{0,1\}^n$ such that both the subproblems (2) created after branching on $(\hat{\pi}, 0)$ are infeasible.*
3. *Given a mathematical program of the form (P), does there exist $\hat{\pi} \in \{0,1\}^n, \hat{\pi}_0 \in \mathbb{Z}$ such that both the subproblems (2) created after branching on $(\hat{\pi}, \hat{\pi}_0)$ are infeasible.*
4. *Given a mathematical program of the form (P), does there exist $\hat{\pi} \in \mathbb{Z}^n_+$ such that both the subproblems (2) created after branching on $(\hat{\pi}, 0)$ are infeasible.*
5. *Given a mathematical program of the form (P), does there exist $\hat{\pi} \in \mathbb{Z}^n$ such that both the subproblems (2) created after branching on $(\hat{\pi}, 0)$ are infeasible.*

6. *Given a mathematical program of the form (P), does there exist $\hat{\pi} \in \{0, 1, -1\}^n, \hat{\pi}_0 \in \mathbb{Z}$ such that both the subproblems (2) created after branching on $(\hat{\pi}, \hat{\pi}_0)$ are infeasible. (This problem is mentioned because Owen and Mehrotra [2001] developed a greedy heuristic for the optimization version of this problem, without addressing the complexity of the problem).*

Proof. The proof of each of the above propositions follows directly from the proof of Proposition 3.2 above. □

If Q is a polytope, then the fact that $Q \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$, for some $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$, is sufficient to show that $w(Q) < 1$. The proof provided above settles the question of complexity of finding such a sufficient condition. If a program of form (P) has only binary variables, i.e., it is of the form:

$$\begin{aligned} \min\ & cx \\ \text{s.t. } & Ax \geq b \\ & x \in \{0,1\}^n, \end{aligned} \qquad (P_b)$$

then the width of the associated polyhedron $\mathcal{P}_b$ is trivially at most one. The following proposition shows that the problem of deciding whether there exists a disjunction $(\hat{\pi}, \hat{\pi}_0)$ that will prove the infeasibility of a binary program is also $\mathcal{NP}$-complete.

Problem 5 (Disjunctive infeasibility for binary programs)**.** *Given a mathematical program of the form (P_b), does there exist a disjunction $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$, that proves infeasibility?*

Problem 5 is a special case of Problem 3 and hence the proof of $\mathcal{NP}-$completeness of the latter follows from that of the former. However, we address the complexity of Problem 5 separately because the proof is easier to understand having seen that of Problem 3.

Proposition 3.4. *Problem 5 is $\mathcal{NP}$-complete.*

Proof. The proof is similar to that of Proposition 3.2. Let $n \in \mathbb{Z}_+$, $S = \{1, 2, \ldots, (n-1)\}, a^i \in \mathbb{Z}_+, i \in S$ be inputs for an instance of Problem 4. We need to modify our previous transformation because coordinates of the feasible region of $\mathcal{P}$ can only lie in $[0, 1]$, while $a^i \in \mathbb{Z}_+, i \in S$. Let $M = \sum_{i \in S} a^i$ and $m = \frac{1}{M}$. If each $a^i, i \in S$ is divided by M, then the problem of partitioning remains the same. Let $\tilde{a}^i (= \frac{a^i}{M}) \in \mathbb{Q}_+, i \in S$ so that $\sum_{i \in S} \tilde{a}^i = 1$. The answer to an instance of Problem 4 is "yes" if and only if there exists a set $K \subseteq \{1, 2, \ldots, n-1\}$ such that $\sum_{i \in K} \tilde{a}^i = \frac{1}{2}$. Since each $\tilde{a}^i$ is an integer multiple of $\frac{1}{M}$, there is no $K \subseteq S$ such that $\sum_{i \in K} \tilde{a}^i \in [\frac{1}{2} - \frac{1}{2M}, \frac{1}{2})$ or $\sum_{i \in K} a^i \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{2M}]$. This observation will be useful later.

Problem 4 can now be reduced to Problem 5 as follows. Let $\epsilon = \frac{1}{2M} = \frac{m}{2}$. Consider the convex hull, $\mathcal{P}_s$, of points $v^i, i = 1 \ldots n+3$ in $n+1$ dimensions, where the coordinates of v^i are defined as

$$v^i_j = \begin{cases} \frac{1}{2n} & \text{if } j \neq i, j \neq n, j \neq n+1, i = 1,2,\ldots,n-1 \\ \frac{1}{2n} + \frac{1}{2} & \text{if } j = i, i = 1,2,\ldots,n-1 \\ 0 & \text{if } j = n, n+1, i = 1,2,\ldots,n-1 \\ & \\ \tilde{a}^j & \text{if } j = 1,2,\ldots,n-1, i = n \\ 1 & \text{if } j = n, n+1, i = n \\ & \\ \tilde{a}^j & \text{if } j = 1,2,\ldots,n-1, i = n+1 \\ \frac{1}{2} - \epsilon & \text{if } j = n, i = n+1 \\ 0 & \text{if } j = n+1, i = n+1 \\ & \\ \tilde{a}^j & \text{if } j = 1,2,\ldots,n-1, i = n+2 \\ 0 & \text{if } j = n, i = n+2 \\ \frac{1}{2} - \epsilon & \text{if } j = n, i = n+2 \\ & \\ \frac{1}{2} & \text{if } j = n, i = n+3 \\ 0 & \text{if } j \neq n, i = n+3 \end{cases}$$

This means $v^1 = (\frac{1}{2n} + \frac{1}{2}, \frac{1}{2n}, \frac{1}{2n}, \ldots, 0, 0)$, $v^2 = (\frac{1}{2n}, \frac{1}{2n} + \frac{1}{2}, \frac{1}{2n}, \frac{1}{2n}, \ldots, 0, 0)$ etc. $v^{n-1} = (\frac{1}{2n}, \ldots, \frac{1}{2n} + \frac{1}{2}, 0, 0)$, $v^n = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{n-1}, 1, 1)$, $v^{n+1} = (\tilde{a}_1, \ldots, \tilde{a}_{n-1}, \frac{1}{2} - \epsilon, 0)$, $v^{n+2} = (\tilde{a}_1, \ldots, \tilde{a}_{n-1}, 0, \frac{1}{2} - \epsilon)$, $v^{n+3} = (0, 0, \ldots, 0, \frac{1}{2}, 0)$. Clearly, $\mathcal{P}_s \subseteq \{x \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, i = 1, 2, \ldots, n\}$. It will now be shown that a $K \subseteq S$ such that $\sum_{i \in K} \tilde{a}^i = \frac{1}{2}$ exists if and only if there exists $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$, such that $\mathcal{P}_s \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$. Suppose $K \subseteq S$ such that $\sum_{i \in K} \tilde{a}^i = \frac{1}{2}$. Let $\hat{\pi}_i = 1, i \in K$, $\hat{\pi}_n = 1, \hat{\pi}_{n+1} = -1$, $\hat{\pi}_i = 0, i \notin K \cup \{n, n+1\}$, $\hat{\pi}_0 = 0$. Then, $0 < \hat{\pi}v^i < 1, i = 1, 2, \ldots, n+3$. Since all vertices of $\mathcal{P}_s$ satisfy the condition $\hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1$, $\mathcal{P}_s \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$. $(\hat{\pi}, \hat{\pi}_0)$ is then the required disjunction.

Conversely, suppose there exists some $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}^{n+1}$ such that $\mathcal{P}_s \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$. Then, $\hat{\pi}_0 < \hat{\pi}v^i < \hat{\pi}_0 + 1, i = 1, 2, \ldots, (n+3)$. This also means that $|\hat{\pi}(v^i - v^k)| < 1, i = 1, 2, \ldots, n-1, k = 1, 2, \ldots, n-1$. Substituting the coordinates of v^i and v^k, one gets: $|\frac{\hat{\pi}_i - \hat{\pi}_k}{2}| < 1$. Because $\hat{\pi}_i, \hat{\pi}_k \in \mathbb{Z}$, this means that $|\hat{\pi}_i - \hat{\pi}_k| \leq 1$ for each pair $(i, k) \in \{1, 2, \ldots, n-1\}^2$. This means that $\hat{\pi}_i \in \{t, t+1\}, i = 1, 2, \ldots, n-1$ for some $t \in \mathbb{Z}$. Let $K = \{i \in S \mid \pi_i = t+1\}$. Substituting the coordinates of v^1 and $\hat{\pi}$ into the inequalities $\hat{\pi}_0 < v^1\hat{\pi} < \hat{\pi}_0 + 1$, one gets:

$$\begin{aligned} \hat{\pi}_0 < \quad & \sum_{i=1}^{n+1} \frac{\hat{\pi}_i}{2n} + \frac{\hat{\pi}_1}{2} < \hat{\pi}_0 + 1 \\ \Rightarrow \hat{\pi}_0 < \frac{t}{2} + & \sum_{i \in K} \frac{1}{2n} + \frac{\hat{\pi}_1}{2} < \hat{\pi}_0 + 1. \end{aligned}$$

Since $\frac{\hat{\pi}_1}{2} \in \{\frac{t}{2}, \frac{t+1}{2}\}$, the only integer value of $\hat{\pi}_0$ that satisfies the above condition is $\hat{\pi}_0 = t$. Thus, $\hat{\pi} \in \{t, t+1\}^n, \hat{\pi}_0 = t$. Also, $K = \phi$ would mean that $\hat{\pi}_0 < t < \hat{\pi}_0 + 1$. This is not possible for any integers $t, \hat{\pi}_0$. Hence, K is not empty. The condition $\hat{\pi}_0 < \hat{\pi} v^n < \hat{\pi}_0 + 1$ implies:

$$\begin{aligned}
\hat{\pi}_0 < \quad & \sum_{i=1}^{n-1} \hat{\pi}_i \tilde{a}^i + \hat{\pi}_n + \hat{\pi}_{n+1} < \hat{\pi}_0 + 1 \\
\Rightarrow \quad t < t & \sum_{i=1}^{n-1} \tilde{a}^i + \sum_{i \in K} \tilde{a}^i + \hat{\pi}_n + \hat{\pi}_{n+1} < t + 1 \\
\Rightarrow \quad & \hat{\pi}_{n+1} = -\hat{\pi}_n
\end{aligned}$$

The condition $\hat{\pi}_0 < \hat{\pi} v^{n+1} < \hat{\pi}_0 + 1$ implies:

$$t < \sum_{i=1}^{n-1} \hat{\pi}_i \tilde{a}^i + \hat{\pi}_n (\frac{1}{2} - \epsilon) < t + 1$$

$$\Rightarrow 0 < \sum_{i \in K} \tilde{a}^i + \hat{\pi}_n (\frac{1}{2} - \epsilon) < 1. \tag{11}$$

The condition $\hat{\pi}_0 < \hat{\pi} v^{n+2} < \hat{\pi}_0 + 1$ implies:

$$t < t + \sum_{i \in K} \tilde{a}^i + \hat{\pi}_{n+1} (\frac{1}{2} - \epsilon) < t + 1$$

$$\Rightarrow 0 < \sum_{i \in K} \tilde{a}^i - \hat{\pi}_n (\frac{1}{2} - \epsilon) < 1. \tag{12}$$

Finally, the condition $\hat{\pi}_0 < \hat{\pi} v^{n+3} < \hat{\pi}_0$ gives:

$$\begin{aligned}
t < \frac{\hat{\pi}_n}{2} < t + 1 \\
\Rightarrow \hat{\pi}_n = 2t + 1.
\end{aligned}$$

Since the disjunction $(\hat{\pi}, \hat{\pi}_0)$ is the same as the disjunction $(-\hat{\pi}, -\hat{\pi}_0 - 1)$, we assume without loss of generality that $\hat{\pi}_n \geq 0$. The condition $\hat{\pi}_n = 2t + 1$ implies that $\hat{\pi}_n \geq 1, t \geq 0$. These, along with the conditions $K \neq \phi, M > 3$, and equation (11) imply that $\hat{\pi}_n = 1$. This along with equations (11, 12) gives,

$$\begin{aligned}
\sum_{i \in K} \tilde{a}^i < \frac{1}{2} + \epsilon \quad \text{and} \\
\sum_{i \in K} \tilde{a}^i > \frac{1}{2} - \epsilon.
\end{aligned}$$

These conditions, along with the choice of ϵ, imply respectively that $\sum_{i \in K} \tilde{a}^i \leq \frac{1}{2}$ and $\sum_{i \in K} \tilde{a}^i \geq \frac{1}{2}$. Therefore, $\sum_{i \in K} \tilde{a}^i = \frac{1}{2}$ and K is the desired subset of S.

To complete the proof, we show that a description of $\mathcal{P}_s$ in the form (P_b) can be obtained in polynomial time from the finite list of points $v^1, v^2, \ldots, v^{n+2}$. Note that the convex hull of $v^1, v^2, \ldots, v^{n+2}$ is a simplex in $(n+1)$ dimensions (say Q), and can be expressed in form (P_b) in time polynomial in the size of the input as follows. If the point $v^{n+3} \in Q$, then $\mathcal{P}_s = Q$. Otherwise, delete from Q any such inequalities that are violated by v^{n+3}. Call this description $\mathcal{P}'$. Consider each of the

$\frac{1}{2}(n+1)(n+2)$ hyperplanes passing through v^{n+3} and any n extreme points of Q. If all of the extreme points of Q lie on one side of this hyperplane, add this to the description $\mathcal{P}'$. Once all such hyperplanes are considered, the region $\mathcal{P}'$ is the same as $\mathcal{P}_s$. This process takes time polynomial in the size of the input and yields a description of $\mathcal{P}_s$ in form (P_b). The proof is now complete. □

The proof provided above is not sufficient to prove a similar result for the restriction of Problem 5 in which $\pi \in \{0,1\}^n$ because one of the components of the vector π in the proof above is restricted to the value of -1. However, the following proof shows that the problem remains $\mathcal{NP}$-complete even in the presence of this restriction. The proof uses a reduction of the ONE-IN-THREE-3SAT problem [Garey and Johnson, 1979; Schaefer, 1978], which is known to be $\mathcal{NP}$-complete, to this problem.

Problem 6 (ONE-IN-THREE-3SAT [Garey and Johnson, 1979])**.** *Given a set U of variables and a collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$ and c does not contain a negated literal. Is there a truth assignment for U such that each clause in C has exactly one true literal?*

Problem 7 (Disjunctive infeasibility of binary programs using 0-1 hyperplanes)**.** *Given a mathematical program of the form (P_b), does there exist $\hat{\pi} \in \{0,1\}^n, \hat{\pi}_0 \in \mathbb{Z}$ such that the feasible region of each LP associated with the subproblems (2) created after branching on $(\hat{\pi}, \hat{\pi}_0)$ (with additional constraints $x \in [0,1]^n$) is empty?*

Proposition 3.5. *Problem 7 is $\mathcal{NP}$-complete.*

Proof. We reduce Problem 6 to Problem 7 as follows. Associate variables $\hat{\pi}_i, i = 1, 2, \ldots, n-1$ with each variable u_i in U (where $n = |U|+1$). Let $\hat{\pi}_i = 1$ if u_i is assigned TRUE in a truth assignment and $\hat{\pi}_i = 0$ otherwise. Clearly, an instance of Problem 6 has a required truth assignment if and only if $\hat{\pi}$ satisfies the following constraints,

$$\sum_{\{i|u_i \in c\}} \hat{\pi}_i = 1, \quad \forall c \in C$$
$$\hat{\pi}_i \in \{0,1\}^n \tag{13}$$

Let $A^{\hat{\pi}}$ be the coefficient matrix associated with the above program (with elements $a_{ij} = 1$ if and only if clause i contains variable u_j, 0 otherwise). If $rank(A^{\hat{\pi}}) < rank(A^{\hat{\pi}}, \mathbf{1})$, then the system (13) is infeasible and there does not exist any truth assignment for Problem 6. Also, any such infeasibility can be detected in polynomial time by calculating the rank of the above matrices. Hence, it may be assumed that $rank(A^{\hat{\pi}}) = rank(A^{\hat{\pi}}, \mathbf{1})$. It may also be assumed that the rows of $A^{\hat{\pi}}$ are linearly independent. Otherwise, one may drop a redundant row from (13) (or equivalently, a redundant clause from Problem 6). Using these facts, one can assume without loss of generality that $|C| = rank(A^{\hat{\pi}}) = rank(A^{\hat{\pi}}, \mathbf{1}) \leq |U| = n$.

Consider the convex hull $\mathcal{P}_s$ of $m = |C|+1$ points: $v^i, i = 1, 2, \ldots m \in \mathbb{R}^n$. Let the coordinate j, v^i_j, of each point v^i assume a value 0 if $a_{ij} = 0$ and a value $\frac{1}{2}$ if $a_{ij} = 1, i = 1, 2, \ldots, m-1, j = 1, 2, \ldots, n$. Let $v^i_n = 0, i = 1, 2, \ldots, m-1$. Let v^m be chosen such that $v^m_j = 0, j = 1, 2, \ldots, n-1, v^m_n = \frac{1}{2}$. There exists a $\hat{\pi} \in \{0,1\}^n$ such that $\mathcal{P}_s \subseteq \{x \in \mathbb{R}^n \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$ if and only if $v^i \in \{x \in \mathbb{R}^n \mid$

$\hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}, i = 1, 2, \ldots, m$. This is true if and only if $(\hat{\pi}, \hat{\pi}_0)$ satisfy the following conditions

$$\begin{aligned} \pi_0 < \frac{1}{2} \sum_{u_j \in c} \hat{\pi}_j < \pi_0 + 1, \quad c \in C \\ \pi_0 < \quad \frac{1}{2} \hat{\pi}_n < \hat{\pi}_0 + 1, \end{aligned}$$

or equivalently, if and only if $(\hat{\pi}, \hat{\pi}_0)$ satisfy the following conditions

$$\begin{aligned} \sum_{u_j \in c} \hat{\pi}_j = 2\pi_0 + 1, \quad c \in C \\ \hat{\pi}_n = 2\hat{\pi}_0 + 1. \end{aligned}$$

Since $\hat{\pi} \in \{0, 1\}^n$, the above conditions are satisfied if and only if $\hat{\pi}_0 = 0, \hat{\pi}_n = 1$ and $\hat{\pi}$ satisfies the system of equations (13). Hence, an instance of Problem 6 has a required truth assignment if and only if $\mathcal{P}_s \subseteq \{x \mid \hat{\pi}_0 < \hat{\pi}x < \hat{\pi}_0 + 1\}$ for some $\hat{\pi} \in \{0, 1\}^n$. Since it was assumed that the rows of $A^{\hat{\pi}}$ are linearly independent, the points v^i are also linearly independent. Hence, the dimension of $\mathcal{P}_s$ is exactly $m - 1 (= |C|)$. In order to obtain a description of $\mathcal{P}_s$ in the standard form (P), one has to find $|C|$ facets of $\mathcal{P}_s$. This can be done by making $|C|$ sets, each with $|C| - 1$ extreme points of $\mathcal{P}_s$ and finding a plane that passes through these. This can be done in time polynomial in the size of the input by solving $|C|$ systems of equations, each in $|C| - 1$ variables. These $|C|$ facets can be used to describe $\mathcal{P}_s$ in standard form (P). Thus, Problem 7 is $\mathcal{NP}$-complete. □

The complexity results for Problem 1 follow directly from those for Problem 3. In particular, Problem 1 is $\mathcal{NP}-$hard and remains so even when the restrictions described in Proposition 3.3 are applied and even for the case of binary programs.

4 Split Inequalities

Valid disjunctions can also be used to generate a broad class of valid inequalities called *split inequalities*. Many classes of valid inequalities have been shown to be special cases of split inequalities (see, for instance, the survey by Cornuéjols [2008]). Given an MILP of the form (P) and the associated polyhedron $\mathcal{P}$, we say that an inequality (α, β) is an elementary split inequality for $\mathcal{P}$ with respect to variables with indices $i = 1, 2, \ldots, d$ if both polyhedra $\{x \in \mathcal{P} \mid \pi x \leq \pi_0\}$ and $\{x \in \mathcal{P} \mid \pi x \geq \pi_0 + 1\}$ (and hence also the union of these polyhedra) are subsets of $\{x \in \mathbb{R}^n \mid \alpha x \geq \beta\}$ for some $\pi \in \mathbb{Z}^d \times 0^{n-d}$ and $\pi_0 \in \mathbb{Z}$. Since (π, π_0) is a valid disjunction for (P), such inequalities are also valid for the problem (P).

The elementary split closure of $\mathcal{P}$ is the region formed by the intersection of $\mathcal{P}$ and all elementary split inequalities of $\mathcal{P}$ with respect to $x_i, i = 1, 2, \ldots, d$. Cook et al. [1990] showed that this closure is also a polyhedron. An inequality is said to have split-rank one if it is valid for the elementary split closure of $\mathcal{P}$ but not for $\mathcal{P}$. Thus all elementary split inequalities have split-rank at most one. Also, any inequality that is a convex combination of two elementary split inequalities has split-rank at most one. However, since two different disjunctions may have been used to generate the two elementary split inequalities thus combined, the convex combination of these inequalities may not necessarily be an elementary split inequality, even though its rank is one. As an example, consider the following system

$$
\begin{aligned}
x_1 &\le 0.8 \\
x_2 &\le 0.8 \\
x &\in \mathbb{Z}^2.
\end{aligned} \tag{14}
$$

The inequality $x_1 + x_2 \le 0$ has split-rank one with respect to (14) because it can be obtained as a convex combination of the elementary split inequalities $x_1 \le 0$ and $x_2 \le 0$. The inequality $x_1 + x_2 \le 0$ separates the points $(\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$, which are in the associated polyhedron $\mathcal{P}$, from the feasible region of (14). If this were an elementary split inequality for $\mathcal{P}$ generated using a disjunction $\pi x \le \pi_0 \vee \pi x \ge \pi_0 + 1$, then the disjunction (π, π_0) should also separate the points $(\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$. So, (π, π_0) should, at least, satisfy the following three constraints:

$$
\begin{aligned}
\pi_0 < \quad & \frac{1}{2}\pi_1 < \pi_0 + 1 \\
\pi_0 < \quad & \frac{1}{2}\pi_2 < \pi_0 + 1 \\
\pi_0 < \quad & \frac{1}{2}(\pi_1 + \pi_2) < \pi_0 + 1 \\
& \pi_0, \pi_1, \pi_2 \in \mathbb{Z}.
\end{aligned} \tag{15}
$$

Since the system (15) is infeasible, there is no such disjunction and hence $x_1 + x_2 \le 0$ is not an elementary split inequality even though it is a convex combination of two such inequalities.

Caprara and Letchford [2003] showed that the problem of determining whether a given point x may be separated from the elementary split closure of $\mathcal{P}$ with respect to the variables $x_i, i = 1, 2, \ldots, d$ is $\mathcal{NP}$-complete. Using the equivalence of separation and optimization, it is easy to see that the problem of deciding whether a given inequality has split rank one or not is $\mathcal{NP}$-complete. However, this does not imply anything about the complexity of showing that a given inequality is an elementary split inequality. We have already seen that even if an inequality has split rank one, it may not be an elementary split inequality. We now show that the complexity of this problem follows directly from Proposition 3.2.

Problem 8. *Given a mathematical program of the form (P), is a given inequality (α, β) an elementary split inequality for $\mathcal{P}$ with respect to the variables $x_i, i = 1, 2, \ldots, d$?*

Proposition 4.1. *Problem 8 is $\mathcal{NP}$-complete*

Proof. Consider the special case when $\alpha = 0, \beta = 1$. Then $\alpha x \ge \beta$ (or $0 \ge 1$) is a split inequality for $\mathcal{P}$ if and only if there exists a disjunction $(\hat{\pi}, \hat{\pi}_0)$ such that the associated LPs (2) are infeasible. By Proposition 3.2, the problem of finding such $(\hat{\pi}, \hat{\pi}_0)$ is $\mathcal{NP}$-complete. □

In contrast, consider the case of Chvátal-Gomory (C-G) inequalities. Given a pure integer program (P) with $d = n$, a C-G inequality for (P) is an inequality of the form

$$\alpha x \ge \lceil \beta \rceil, \tag{16}$$

where $\alpha \in \mathbb{Z}^n$ and $\alpha x \ge \beta$ is a valid inequality for the feasible region $\mathcal{P}$ of the LP relaxation. Eisenbrand [1999] showed that the problem of separating a given point x from the elementary C-G closure of (P) is $\mathcal{NP}$-complete. Hence, the problem of deciding whether a given inequality has C-G rank one or not is also, like for the case of split inequalities, $\mathcal{NP}$-complete. However, unlike split inequalities, we can decide whether a given inequality (say $\alpha x \geq \delta$, where $\alpha \in \mathbb{Z}^n, \delta \in \mathbb{Z}$) is an elementary C-G inequality or not in polynomial time by solving the LP

$$\begin{aligned} \min\ & \alpha x \\ & Ax \geq b. \end{aligned} \tag{17}$$

Then, $\alpha x \geq \delta$ is an elementary C-G inequality for (P) if and only if the optimal solution to (17) is strictly greater than $\delta - 1$.

The above results seem somewhat surprising. On the one hand, the problem of deciding whether the rank of a given inequality is one lies in the complexity class $\mathcal{NP}$-complete for the case of both C-G inequalities and split inequalities. On the other hand, the problem of deciding whether a given inequality is an elementary C-G inequality lies in complexity class $\mathcal{P}$, while the same problem for a split inequality lies in complexity class $\mathcal{NP}$-complete.

We now consider the following problem of increasing the LP relaxation bound by using split inequalities.

Problem 9. *Given a mathematical program of the form (P) and $z_l \in \mathbb{R}$, does there exist a set S of split inequalities of rank one associated with $\mathcal{P}$ such that the LP relaxation bound achieved after adding all of the inequalities in S is at least z_l?*

Problem 9 is equivalent to optimizing over the elementary closure associated with (P) and hence is $\mathcal{NP}-$complete. Now consider a related problem.

Problem 10. *Given a mathematical program of the form (P) and $z_l \in \mathbb{R}$, does there exist a single elementary split inequality for (P) such that the LP relaxation bound achieved after adding it is at least z_l?*

Proposition 4.2. *Problem 10 is $\mathcal{NP}$-complete.*

Proof. Problem 10 can be shown to be equivalent to Problem 3. Suppose there is a disjunction, say $(\hat{\pi}, \hat{\pi}_0)$, such that the LP objective function value associated with each subproblem is at least z_l, i.e., $(\hat{\pi}, \hat{\pi}_0)$ is a solution to Problem 3. Then, $cx \geq z_l$ is a valid inequality for both $\mathcal{P} \cap \{x \mid \hat{\pi}x \leq \hat{\pi}_0\}$ and $\mathcal{P} \cap \{x \mid \hat{\pi}x \geq \hat{\pi}_0 + 1\}$. Therefore, $cx \geq z_l$ is a valid elementary split inequality that when added to the LP relaxation makes the objective function value at least z_l.

Conversely, suppose there exists an elementary split inequality, say (α, β), derived from a disjunction $(\hat{\pi}, \hat{\pi}_0)$, such that the LP bound for the polytope $\mathcal{P}^1 = \mathcal{P} \cap \{x|\alpha x \geq \beta\}$ is at least z_l. Then $cx \geq z_l$ is a valid inequality for $\mathcal{P}^1$. This means that $cx \geq z_l$ is a valid inequality for the two subsets of $\mathcal{P}^1 : \mathcal{P} \cup \{x|\hat{\pi}x \leq \hat{\pi}_0\}, \mathcal{P} \cup \{x|\hat{\pi}x \geq \hat{\pi}_0 + 1\}$ and hence branching on the disjunction $(\hat{\pi}, \hat{\pi}_0)$ will push the LP bound to at least z_l.

□

The equivalence of Problem 10 and Problem 3 has been noted previously by, among others, Karamanov and Cornuéjols [2007]. Nevertheless, we provide the proof above for completeness. It is also well known that one can always find an elementary split inequality (for instance, a Gomory mixed integer inequality) to separate a given basic feasible solution of the LP relaxation, that is not feasible for the original problem, from the elementary split closure in time polynomial in size of the input (see, for instance, Cornuéjols [2008]). However, the above problem of maximizing the lower bound by adding a valid split inequality remains $\mathcal{NP}-$hard even if an optimal basic feasible point of $\mathcal{P}$ is provided as an input because the desired inequality must separate the set of all points that have lower objective values regardless of whether they are basic feasible solutions.

5 Conclusions

In this paper, we showed that the problem of selecting a general branching disjunction so that the LP relaxation associated with each subproblem becomes infeasible is $\mathcal{NP}$-complete. This leads to two important results—that the problem of selecting a general branching disjunction that maximizes the bound improvement of a given MILP is $\mathcal{NP}$-hard, and that the problem of deciding whether a given inequality is an elementary split inequality is $\mathcal{NP}$-complete. We also showed that the former problem remains $\mathcal{NP}$-hard even when several natural restrictions are imposed on the disjunctions or when all integer-constrained variables in the MILP are binary. These complexity results and the results from computational experiments, as described in [Mahajan and Ralphs, 2009], provide motivation for developing fast heuristics to solve these problems so that good disjunctions may be obtained in reasonable time. A disjunction used for branching could alternatively be used also for generation of an associated split inequality. We used this fact to prove some results for problems associated with generating split inequalities. Although some studies focused on understanding the relationship between these two alternatives have already been carried out, this topic deserves more attention.

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