Comparing Nonconvex Programs:
Discrete and Continuous

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Overview

Introduction to RLT-based techniques
- A lift-and-project tightening for linear 0-1 programs
- Can I get the convex hull?

Quadratic Programs: Discrete (0-1) Variables
- Constructing a higher-dimensional LP representation
- Generating RLT constraints: Bound-factor products
- A branch-and-bound algorithm

Quadratic Programs: Continuous Variables
- Constructing a higher-dimensional LP representation
- A branch-and-bound algorithm
- Comparing the discrete and continuous cases

Decision Tree Analysis
- Path-based formulation
- Discrete and Continuous Representations
Introduction to RLT: A Linear 0-1 Example
Solving MINLPs using LPs!

- 0-1 programs to continuous (nonconvex) factorable programs
- DC or DM optimization, branch-and-reduce, *etc.*
- BARON, OQNLP, LGO, *etc.*

**Reformulation – Linearization Technique (RLT)**

**Basic idea:**
Easiest optimization problems to solve are linear programs (LPs)
Take advantage to construct tight higher-dimensional LP representations to a given nonconvex program
Derive other valid inequalities to strengthen the RLT relaxation
Embed the RLT relaxation into a (convergent) branch-and-bound process
Consider the following linear 0-1 program:

Maximize \( 3x_1 + 4x_2 \)

s.t. 

\( 4x_1 + 3x_2 \leq 6 \)

\((x_1, x_2) \in \{0, 1\}^2\)

Optimum \((1, 2/3): z_{LP} = 25/4\)

Optimum \((3/4, 1): z_{LP} = 17/4\)

RLT constraints

\( -2x_1 + 3w_{12} \leq 0 \)

\( 2x_1 + x_2 - w_{12} \leq 2 \)

\( -3x_2 + 4w_{12} \leq 0 \)

\( 2x_1 + 3x_2 - 2w_{12} \leq 2 \)

Hmm... does RLT really help?
Hmm…does RLT really help? (contd.)

How about going one step further?

\[ 2x_1 + 3x_2 \leq 3 \quad \leftarrow \quad \times (1 - x_1) \]
\[ \Rightarrow \quad 3x_2 - 3w_{12} \leq 3 - 3x_1 \]
\[ \Rightarrow \quad x_1 + x_2 - w_{12} \leq 1 \]

We have generated the Convex Hull 😊!

\[ (1 - x_1)(1 - x_2) \geq 0 \]
\[ \Rightarrow \quad x_1 + x_2 - w_{12} \leq 1 \]
Quadratic Programs: Discrete (0–1) Variables
Minimize \[ \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij} x_i x_j \]
subject to \[ \sum_{i=1}^{n} a_{ki} x_i \leq b_k, \forall k \]
\[ x_i \in \{0, 1\} \]

**RLT VARIABLES**
\[ y_{ij} = x_i x_j \]

**RLT CONSTRAINTS**
\[ y_{ij} \leq x_i, \ \forall i \]
\[ y_{ij} \leq x_j, \ \forall j \]
\[ -y_{ij} + x_i + x_j \leq 1, \ \forall (i, j) \]
\[ y \geq 0 \]

**LP relaxation:** \((\bar{x}, \bar{y})\);  **Branching Strategy:** \( \theta_{uv} = \arg \max_{(i, j)} \left\{ |\bar{y}_{ij} - \bar{x}_i \bar{x}_j| \right\} \)

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Bound-Factor Products

\[ (x_i - 0) \quad (1-x_i) \]
\[ (x_j - 0) \quad (1-x_j) \]

\[ e.g. \quad x_i (1-x_j) \geq 0 \]
\[ \Rightarrow x_i - x_i x_j \geq 0 \quad \Rightarrow x_i - y_{ij} \geq 0. \]

\[ e.g. \quad (1-x_i)(1-x_j) \geq 0 \]
\[ \Rightarrow 1 - x_i - x_j + x_i x_j \geq 0 \]
\[ \Rightarrow -x_i - x_j + y_{ij} \geq -1. \]
Pure 0-1 QPs (contd.)

Minimize \( f(x_1, x_2) = x_1 + x_1^2 - 2x_2 - x_2^2 - 5x_1x_2 \)
subject to:

\[
\begin{align*}
4x_1 + 2x_2 &\leq 5 \\
-2x_1 + 5x_2 &\leq 4
\end{align*}
\]
\( x_1; x_2 \)
\( (1-x_1); (1-x_2) \)
\( x_1, x_2 \in \{0,1\}. \)

\[
\begin{align*}
e.g. & \quad (4x_1 + 2x_2 - 5)(1-x_2) \leq 0 \\
\Rightarrow & \quad 4x_1 + 2x_2 - 5 - x_2(4x_1 + 2x_2 - 5) \leq 0 \\
\Rightarrow & \quad 4x_1 + 2x_2 - 5 - 4w_{12} - 2x_2 + 5x_2 \leq 0 \\
\Rightarrow & \quad 4x_1 + 5x_2 - 4w_{12} \leq 5
\end{align*}
\]
RLT Relaxation

Minimize \( f(x_1, x_2, w_{12}) = 2x_1 - 3x_2 - 5w_{12} \)

subject to:

- \(- x_1 + 2w_{12} \leq 0\)
- \(-3x_2 + 4w_{12} \leq 0\)
- \(-6x_1 + 5w_{12} \leq 0\)
- \(x_2 - 2w_{12} \leq 0\)

\(5x_1 + 2x_2 - 2w_{12} \leq 5\)
\(4x_1 + 5x_2 - 4w_{12} \leq 5\)
\(4x_1 + 5x_2 - 5w_{12} \leq 4\)
\(-2x_1 + 4x_2 + 2w_{12} \leq 4\)

\(w_{12} \leq x_1\)
\(w_{12} \leq x_2\)

\(-w_{12} + x_1 + x_2 \leq 1\)

\(x_1, x_2 \in \{0, 1\}; \quad w_{12} \geq 0.\)
Remark: The LP relaxation without Type 1 and 2 constraints is $z^* = -5.25$.

$$z_{LP}^* = -2.154$$
$$\bar{x}_1 = 0.615; \quad \bar{x}_2 = 0.615$$
$$\bar{w}_{12} = 0.308$$

$$|\bar{w}_{12} - \bar{x}_1 \bar{x}_2| = 0.0702$$

$x_1 = 0$

$$z^* = 0$$
$$x_1^* = 0; \ x_2^* = 0$$
$$w_{12}^* = 0$$

optimum

$x_1 = 1$

$$z^* = 2.0$$
$$x_1^* = 1.0; \ x_2^* = 0$$
$$w_{12}^* = 0$$

fathom
Initialization
(Input Data for NCP)
Set active node = 1

Select an active node and construct RLT Relaxation. If no active node, incumbent = optimum, done!

Solve LP Relaxation

Is solution binary?

Fathom node and update incumbent. Update active node set

LP relaxation > incumbent?

Select branching variable, remove current node, add two children nodes to active set
Continuous Nonconvex Programs

Minimize \( f(x_1, x_2) = x_1 + x_1^2 - 2x_2 - x_2^2 - 5x_1x_2 \)
subject to:
\[
4x_1 + 2x_2 \leq 5 \\
-2x_1 + 5x_2 \leq 4 \\
0 \leq x_1, x_2 \leq 1.
\]

\begin{align*}
e.g. \quad & (1 - x_1)(1 - x_2) \geq 0 \\
\Rightarrow \quad & -x_1 - x_2 + w_{12} \leq 1 \\
e.g. \quad & (4x_1 + 2x_2 - 5)(1 - x_2) \leq 0 \\
\Rightarrow \quad & 4x_1 + 2x_2 - 5 - 4x_1x_2 - 2x_2^2 + 5x_2 \leq 0 \\
\Rightarrow \quad & 4x_1 + 7x_2 - 4w_{12} - 2w_{22} \leq 5
\end{align*}
RLT Relaxation (Shortened)

Minimize \( f(x_1, x_2) = x_1 + w_{11} - 2x_2 - w_{22} - 5w_{12} \)

subject to:

\[
\begin{align*}
4x_1 + 2x_2 & \leq 5 \\
-2x_1 + 5x_2 & \leq 4 \\
x_1 & \leq w_{11} \\
x_1 & \leq w_{12} \\
x_2 & \leq w_{12} \\
x_2 & \leq w_{22} \\
x_1 - w_{11} & \leq 1 \\
x_2 - w_{22} & \leq 1 \\
x_1 + x_2 - w_{12} & \leq 1 \\
0 & \leq x_1, x_2, x_3, w_{11}, w_{12}, w_{22} \leq 1
\end{align*}
\]

CONSTRANT-BOUND FACTOR PRODUCTS NOT INCLUDED IN THIS RELAXATION!
Branch-and-Bound Tree

\[ z_{LP}^* = -6.0 \]
\[ \bar{x}_1 = 0.75; \bar{x}_2 = 1.0; \]
\[ \bar{w}_{11} = 0.5; \bar{w}_{12} = 0.75; \bar{w}_{22} = 1.0 \]

\[ x_1 \leq 0.75 \]

\[ (x_1 - 0) (0.75 - x_1) \]
\[ (x_2 - 0) (1 - x_2) \]
\[ e.g. \quad (0.75 - x_1)(1 - x_2) \geq 0 \]
\[ \Rightarrow \quad x_1 + 0.75x_2 - w_{12} \leq 0.75 \]

\[ x_1 \geq 0.75 \]

\[ (x_1 - 0.75) (1 - x_1) \]
\[ (x_2 - 0) (1 - x_2) \]
\[ e.g. \quad (x_1 - 0.75)(1 - x_2) \geq 0 \]
\[ \Rightarrow \quad x_1 + 0.75x_2 - w_{12} \geq 0.75 \]

\[ |\bar{w}_{11} - \bar{x}_1^2| = 0.0702 \]
\[ \bar{x}_1 = 0.75; \bar{x}_2 = 1.0 \]
\[ z_{UB} = -5.4375 \]
Branch-and-Bound Tree (contd.)

\[ z_{\text{LP}}^* = -6.0 \]
\[ x_1 = 0.75; x_2 = 1.0; \]
\[ w_{11} = 0.5; w_{12} = 0.75; w_{22} = 1.0 \]
\[ x_1 = 0.75; x_2 = 1.0 \]
\[ z_{\text{UB}} = -5.4375 \]
\[ x_1 \leq 0.75 \]
\[ x_1 \geq 0.75 \]

\[ z_{\text{LP}}^* = -5.4375 \]
\[ x_1 = 0.75; x_2 = 1.0; \]
\[ w_{11} = 0.5625; w_{12} = 0.75; w_{22} = 1.0 \]
\[ z_{\text{LP}}^* = -5.4375 \]
\[ x_1 = 0.75; x_2 = 1.0; \]
\[ w_{11} = 0.5625; w_{12} = 0.75; w_{22} = 1.0 \]
Solving Discrete Quadratic Programs Using Continuous Variables
Minimize \( f(x_1, x_2) = x_1 + x_1^2 - 2x_2 - x_2^2 - 5x_1x_2 \)
subject to:
\[
\begin{align*}
4x_1 + 2x_2 & \leq 5 \\
-2x_1 + 5x_2 & \leq 4 \\
x_1, x_2 & \in \{0, 1\}.
\end{align*}
\]

A Continuous Representation

\[
f(x_1, x_2, w_{11}, w_{12}, w_{22}) = x_1 + w_{11} - 2x_2 - w_{22} - 5w_{12}
\]

\[
\begin{align*}
4x_1 + 2x_2 & \leq 5 & w_{12} & \leq x_1 \\
-2x_1 + 5x_2 & \leq 4 & w_{12} & \leq x_2 \\
x_1 - w_{11} &= 0 & -w_{12} + x_1 + x_2 & \leq 1 \\
x_2 - w_{22} &= 0 & 0 \leq x_1, x_2, w_{11}, w_{12}, w_{22} & \leq 1.
\end{align*}
\]
A Continuous Representation (contd.)

\[ z_{LP}^* = -5.25 \]

\[
\begin{align*}
\bar{x}_1 &= 0.75; \quad \bar{x}_2 = 1.0; \\
\bar{w}_{11} &= 0.75; \quad \bar{w}_{12} = 0.75; \quad \bar{w}_{22} = 1.0
\end{align*}
\]

\[ z_{LP}^* = -2.1538 \]

\[
\begin{align*}
\bar{x}_1 &= 0.615; \quad \bar{x}_2 = 0.615; \\
\bar{w}_{11} &= 0.615; \quad \bar{w}_{12} = 0.308; \quad \bar{w}_{22} = 0.615
\end{align*}
\]
Summary

- Looked at constructing RLT relaxations for quadratic discrete (0-1) programs and quadratic programs defined in terms of continuous variables
- Branch-and-bound algorithms for both cases
- A continuous representation for quadratic 0-1 problems
- Comparison of relaxations
- Ideas for further talks include insights into developing relaxations for higher order polynomial programs, tightening the RLT relaxation using semidefinite cuts, and applications.

Questions?
Thanks...