Computational testing of exact separation for mixed-integer knapsack problems

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MIP 2008 - Columbia University
MIP solvers & cutting planes

MIP solvers include cut generation routines looking at single-row relaxations:

- Knapsack ⇒ Lifted Cover Inequalities
- Mixed knapsack ⇒ Mixed-Integer Rounding (MIR) inequalities
- Tableau rows ⇒ Gomory Cuts
MIP solvers & cutting planes

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“Cuts outside the template paradigm”

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- Local cuts proved to be successful for the TSP
“Cuts outside the template paradigm”

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- We can try to generate “cuts outside the template paradigm” (local cuts: Applegate, Bixby, Chvátal and Cook, 2000)

- Local cuts proved to be successful for the TSP

- Based on exact separation.
Exact separation

- Given: a polyhedron $P \subset \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$. 
Exact separation

- Given: a polyhedron \( P \subset \mathbb{R}^n \) and a point \( \bar{x} \in \mathbb{R}^n \).

- A separation algorithm is said exact if it either guarantees to provide a valid inequality for \( P \) cutting off \( \bar{x} \) or concludes that \( \bar{x} \in P \).
Exact separation of valid inequalities for the knapsack polytope

The knapsack set (Boyd, 1988)

\[ X^K = \{ y \in \mathbb{Z}_+^n : ay \leq b, \ y \leq u \} \]
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The exact separation LP $SEPLP(X^K)$:

\[
\begin{align*}
\text{max} & \quad \bar{y}\pi - \pi_0 \\
\text{s.t.} & \quad w\pi \leq \pi_0, \quad w \in X^K \\
& \quad 1\pi = 1 \\
& \quad \pi, \pi_0 \geq 0
\end{align*}
\]

(1) (2)
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\( \bar{y} \in \mathbb{R}^n \) is the fractional point to cut-off.
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Inequalities (1) ensure that the inequality is satisfied from every feasible solution in \( X^K \).
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(2) is a normalization constraint.
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Let \( \pi^*, \pi_0^* \) be the optimal solution of \( SEPLP(X^K) \).
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The inequality \( \pi^* y \leq \pi_0^* \) is valid for \( \text{conv}(X^K) \).
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Extreme points of \( SEPLP(X^K) \) are in one-to-one correspondence with the facets of \( \text{conv}(X^K) \).
Recent results

Extension of the “local cuts” technique to MIP problems

- Espinoza (2006)
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MIPLIB instances

- Kaparis and Letchford (2007) yielded tighter lower bounds for several MIPLIB instances
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Generalized Assignment problem

- Medium-size Generalized Assignment instances $d10200$ and $d20200$ solved to optimality for the first time.
- Integrality gap reduced on many larger benchmark instances (up to $80 \times 1600$) (A., Boccia and Vasilyev, 2007).
Recent results (cont.)

Single Source Capacitated Facility Location Problems

- Reformulation based on dicut inequalities + exact separation (Boccia, 2007).
- Many benchmark instances solved to optimality (MIP solvers failed).
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Set Covering

- Exact separation for subsets of formulation constraints (A., Boccia and Vasyliev, 2007).
- seymour solved to optimality on a single workstation.
A step further: the mixed-integer knapsack set $X^{MI}$

We consider single-row mixed-integer knapsack relaxations of MIP problems:

$$X^{MI} = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : ay + gx \leq b, y \leq u, x \leq v \}$$
A step further: the mixed-integer knapsack set $X^{MI}$

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$$X^{MI} = \{(y, x) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+: ay + gx \leq b, y \leq u, x \leq v\}$$

- Atamturk (2002) studied the polyhedral structure of $\text{conv}(X^{MI})$. 
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- Atamturk (2002) studied the polyhedral structure of $\text{conv}(X^{MI})$.

- Fukasawa and Goycoolea (2007) proposed an exact separation routine for $X^{MI}$. The core of their separation procedure is a sophisticated Branch-and-Bound algorithm for the mixed-integer knapsack problem.
The knapsack set with a single continuous variable $X^{MK}$

If in

$$X^{Ml} = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : ay + gx \leq b, y \leq u, x \leq v\}$$

we remove bounds $v$ and aggregate the continuous variables we get the “weaker” knapsack set with a single continuous variable $X^{MK}$:

$$X^{MK} = \{(y, s) \in \mathbb{Z}_+^n \times \mathbb{R}_+ : ay - s \leq b, y \leq u\}$$
The knapsack set with a single continuous variable $X^{MK}$

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Why we focus on $X^{MK}$

The set $X^{MK}$ is a better candidate for a “lightweight” exact separation routine.
A few remarks on $\text{conv}(X^{MK})$

- The polyhedron $\text{conv}(X^{MK})$ was investigated by Marchand and Wolsey (1999)
A few remarks on \( \text{conv}(X^{MK}) \)

- The polyhedron \( \text{conv}(X^{MK}) \) was investigated by Marchand and Wolsey (1999)

- They showed that Mixed-Integer Rounding (MIR) inequalities

\[
\sum_{j=1}^{n} \left( \left\lfloor a_j \right\rfloor + \frac{(f_{a_j} - f_b)^+}{1 - f_b} \right) x_j \leq \lfloor b \rfloor + \frac{s}{1 - f_b}
\]

(where \( f_d = d - \lfloor d \rfloor \)) can be easily derived from \( X^{MK} \).
A few remarks on $\text{conv}(X^{MK})$

- The polyhedron $\text{conv}(X^{MK})$ was investigated by Marchand and Wolsey (1999)

- They showed that Mixed-Integer Rounding (MIR) inequalities

$$\sum_{j=1}^{n} \left( \lfloor a_j \rfloor + \frac{(f_{a_j} - f_b)^+}{1 - f_b} \right) x_j \leq \lfloor b \rfloor + \frac{s}{1 - f_b}$$

(where $f_d = d - \lfloor d \rfloor$) can be easily derived from $X^{MK}$.

- They characterized several other classes of valid inequalities for $\text{conv}(X^{MK})$
Exact separation for $\text{conv}(X^{MK})$

Any valid inequality for $\text{conv}(X^{MK})$ has the form:

$$\pi y - \sigma s \leq \pi_0,$$

with $\pi$, $\sigma$ and $\pi_0$ nonnegative.
Exact separation for $\text{conv}(X^{MK})$

Any valid inequality for $\text{conv}(X^{MK})$ has the form:

$$\pi y - \sigma s \leq \pi_0,$$

with $\pi$, $\sigma$ and $\pi_0$ nonnegative.

Solve $\text{SEPLP}(X^{MK})$:

$$\max \quad \bar{y}\pi - \bar{s}\sigma - \pi_0$$

$$w\pi - t\sigma \leq \pi_0, \quad (w, t) \in X^{MK} \quad (3)$$

$$1\pi + \sigma = 1 \quad (4)$$

$$\pi, \sigma, \pi_0 \geq 0$$
Exact separation for $\text{conv}(X^{MK})$

Any valid inequality for $\text{conv}(X^{MK})$ has the form:

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$$\max \quad \bar{y} \pi - \bar{s} \sigma - \pi_0$$

$$\begin{align*}
  \mathbf{w} \pi - t \sigma & \leq \pi_0, \quad (\mathbf{w}, t) \in X^{MK} \\
  \mathbf{1} \pi + \sigma & = 1 \\
  \pi, \sigma, \pi_0 & \geq 0
\end{align*}$$

$(\bar{y}, \bar{s}) \in \mathbb{R}^n$ is the fractional point to cut-off.
Exact separation for \( \text{conv}(X^{MK}) \)

Any valid inequality for \( \text{conv}(X^{MK}) \) has the form:

\[ \pi y - \sigma s \leq \pi_0, \]

with \( \pi, \sigma \) and \( \pi_0 \) nonnegative.

Solve \( \text{SEPLP}(X^{MK}) \):

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\begin{align*}
\max & \quad \bar{y}\pi - \bar{s}\sigma - \pi_0 \\
\text{subject to} & \quad w\pi - t\sigma \leq \pi_0, \quad (w, t) \in X^{MK} \quad (3) \\
& \quad 1\pi + \sigma = 1 \quad (4) \\
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Inequalities (3) ensure that the inequality is satisfied from every feasible solution in \( X^{MK} \).
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Solve $SEPLP(X^{MK})$:

$$\max \hspace{1em} \bar{y} \pi - \bar{s} \sigma - \pi_0$$

$$w \pi - t \sigma \leq \pi_0, \hspace{1em} (w, t) \in X^{MK}$$

$$1 \pi + \sigma = 1$$

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(4) is a normalization constraint.
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Let $\pi^*$, $\sigma^*$, $\pi_0^*$ be the optimal solution of $\text{SEPLP}(X^{MK})$. 
Exact separation for \( \text{conv}(X^{MK}) \)

Any valid inequality for \( \text{conv}(X^{MK}) \) has the form:

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\pi \mathbf{y} - \sigma \mathbf{s} \leq \pi_0,
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with \( \pi, \sigma \) and \( \pi_0 \) nonnegative.

**Solve \( \text{SEPLP}(X^{MK}) \):**

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\begin{align*}
\max & \quad \bar{y} \pi - \bar{s} \sigma - \pi_0 \\
\text{subject to} & \quad \mathbf{w} \pi - t \sigma \leq \pi_0, \quad (\mathbf{w}, t) \in X^{MK} \\
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\end{align*}
\]

The inequality \( \pi^* \mathbf{y} - \sigma^* \mathbf{s} \leq \pi_0^* \) is valid for \( \text{conv}(X^{MK}) \).
Exact separation for $\text{conv}(X^{MK})$

Any valid inequality for $\text{conv}(X^{MK})$ has the form:

$$\pi y - \sigma s \leq \pi_0,$$

with $\pi$, $\sigma$ and $\pi_0$ nonnegative.

Solve $\text{SEPLP}(X^{MK})$:

$$\max \quad \bar{y}\pi - \bar{s}\sigma - \pi_0$$

$$w\pi - t\sigma \leq \pi_0, \quad (w, t) \in X^{MK} \quad (3)$$

$$1\pi + \sigma = 1 \quad (4)$$

$$\pi, \sigma, \pi_0 \geq 0$$

Extreme points of $\text{SEPLP}(X^{MK})$ are in one-to-one correspondence with the facets of $\text{conv}(X^{MK})$. 
Solving $SEPLP(X^{MK})$ by row generation

Step 1 Let $S \subset X^{MK}$ be a subset of the feasible solutions.
Solving $SEPLP(X^{MK})$ by row generation

Step 1 Let $S \subset X^{MK}$ be a subset of the feasible solutions.

Step 2 Solve the partial separation problem $SEPLP(S)$:

$$\begin{align*}
\max & \quad \bar{y}\pi - \bar{s}\sigma - \pi_0 \\
\text{subject to} & \quad w\pi - t\sigma \leq \pi_0, \quad (w, t) \in S \\
& \quad \pi + \sigma = 1 \\
& \quad \pi, \pi_0 \geq 0
\end{align*}$$

Let $(\pi^*, \sigma^*, \bar{\pi}_0^*)$ be the optimal solution of $SEPLP(S)$. 
Solving $SEPLP(X^{MK})$ by row generation

Step 1 Let $S \subset X^{MK}$ be a subset of the feasible solutions.

Step 2 Solve the *partial separation* problem $SEPLP(S)$:

$$\begin{align*}
\text{max} & \quad \bar{y} \pi - \bar{s} \sigma - \pi_0 \\
\text{subject to} & \quad w \pi - t \sigma \leq \pi_0, \quad (w, t) \in S \\
& \quad \pi + \sigma = 1 \\
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\end{align*}$$

Let $(\pi^*, \sigma^*, \bar{\pi}_0^*)$ be the optimal solution of $SEPLP(S)$.

Step 3 Solve the mixed-integer knapsack problem $MKNAP$

$$\begin{align*}
\text{max} & \quad \pi^* w - \sigma^* t \\
\text{subject to} & \quad (w, t) \in X^{MK}
\end{align*}$$

to check whether the “candidate inequality” $\pi^* y - \sigma^* s \leq \pi_0^*$ is valid for $\text{conv}(X^{MK})$. 
Step 4  Let \((\hat{w}, \hat{t})\) be the optimal solution of \(MKNAP\). If \(\pi^* \hat{w} - \sigma^* \hat{t} > \pi_0^*\) then set \(S = S \cup \{(\hat{w}, \hat{t})\}\) and goto Step 1.
Solving $SEPLP(X^{MK})$ by row generation (cont.)

Step 4 Let $(\hat{w}, \hat{t})$ be the optimal solution of $MKNAP$. If $\pi^* \hat{w} - \sigma^* \hat{t} > \pi_0^*$ then set $S = S \cup \{(\hat{w}, \hat{t})\}$ and goto Step 1.

Step 5 $(\pi^*, \sigma^*, \pi_0^*)$ is the optimal solution of $SEPLP(X^{MK})$ and the inequality $\pi^* y - \sigma^* s \leq \pi_0^*$ is valid for $conv(X^{MK})$. 
Solving \textit{MKNAP} efficiently
Solving *MKNAP* efficiently

- The mixed-integer knapsack problem *MKNAP*:

\[
\begin{align*}
\text{max} & \quad \pi^* w - \sigma^* t \\
aw - t & \leq b \\
w & \in \mathbb{Z}^n \\
t & \geq 0
\end{align*}
\]

must be solved repeatedly.
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- The mixed-integer knapsack problem *MKNAP*:
  
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  \text{where} \quad & w \in \mathbb{Z}^n \\
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  must be solved repeatedly.

- We need a very efficient algorithm.
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must be solved repeatedly.

- We need a very efficient algorithm.

**Proposition**

*For any optimal solution* \((\hat{w}, \hat{t})\) of *MKNAP* we have \(\hat{t} = \max(0, a\hat{w} - b)\).
Solving *MKNAP* efficiently

- The mixed-integer knapsack problem *MKNAP*:

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\begin{align*}
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must be solved repeatedly.

- We need a very efficient algorithm.

**Proposition**

*For any optimal solution* \((\hat{w}, \hat{t})\) *of MKNAP we have*

\[
\hat{t} = \max(0, a\hat{w} - b).
\]

*It follows that:*

\[
(\hat{t} = 0) \lor (\hat{t} = a\hat{w} - b > 0)
\]
Solving *MKNAP* efficiently (cont.)

**Proposition**

*The optimal solution of MKNAP is the best between the optimal solutions of the two following knapsack problems:*

**KNAP1** (*t* = 0):

\[
\max \pi^* w \\
w^a w \leq b \\
w \in \mathbb{Z}^n
\]

**KNAP2** (*t* = *aw* − *b*):

\[
\min (\bar{\sigma}^* a - \pi^*) \\
w^a w \geq b + 1 \\
w \in \mathbb{Z}^n
\]

Both the knapsack problems can be solved very fast by dynamic programming (Pisinger, 2004).
Proposition

The optimal solution of MKNAP is the best between the optimal solutions of the two following knapsack problems:

KNAP1 \((t = 0)\):

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Solving \textit{MKNAP} efficiently (cont.)

**Proposition**

\textit{The optimal solution of MKNAP is the best between the optimal solutions of the two following knapsack problems:}

\textbf{KNAP1} \hspace{1em} (t = 0):

\[ \max \pi^* w \\
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\textbf{KNAP2} \hspace{1em} (t = aw - b):

\[ \min (\sigma^* a - \pi^*)w \\
aw \geq b + 1 \\
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Proposition

The optimal solution of MKNAP is the best between the optimal solutions of the two following knapsack problems:

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**KNAP2** \((t = aw - b)\):

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\min (\tilde{\sigma}^* a - \pi^*) w \\
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\]

Both the knapsack problems can be solved very fast by dynamic programming (Pisinger, 2004).
When embedded into a cutting plane algorithm, \( SEPLP(X^{MK}) \) is applied to each row defining a mixed-integer knapsack set:

\[
X^{MI} = \{ (y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : ay + gx \leq b, \ y \leq u, \ x \leq v \}.
\]

Some operations are required to put the row in the “right” form:
Implementation details

When embedded into a cutting plane algorithm, $SEPLP(X^{MK})$ is applied to each row defining a mixed-integer knapsack set:

$$X^{Mi} = \{(y, x) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : ay + gx \leq b, \ y \leq u, \ x \leq v\}.$$

Some operations are required to put the row in the “right” form:

Bound substitution: replace a subset of continuous variable by their simple or variable bounds.
Implementation details

When embedded into a cutting plane algorithm, $SEPLP(X^{MK})$ is applied to each row defining a mixed-integer knapsack set:

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Some operations are required to put the row in the “right” form:

**Bound substitution:** replace a subset of continuous variable by their simple or variable bounds.

**Preprocessing:** transform the mixed integer set $X^{MI}$ into the mixed-integer knapsack set $X^{MK}$. 
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**Bound substitution:** replace a subset of continuous variable by their simple or variable bounds.

**Preprocessing:** transform the mixed integer set $X^{MI}$ into the mixed-integer knapsack set $X^{MK}$.

**Convert coefficients into integers** (required to use dynamic programming)
Bound substitution

- Consider the mixed-integer set

\[ X^{MI} = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : ay + gx \leq b, \ y \leq u, lx \leq v\}. \]
Bound substitution

Consider the mixed-integer set

\[ X^M = \{(y, x) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : \ ay + gx \leq b, \ y \leq u, lx \leq v \}. \]

The MIP formulation can also include some additional variable bounds on the continuous variables.
Bound substitution

Consider the mixed-integer set

\[ X^M = \{ (y, x) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : ay + gx \leq b, \ y \leq u, lx \leq v \}. \]

The MIP formulation can also include some additional variable bounds on the continuous variables.

Bound substitution consists of replacing some continuous variables by their respective simple/variable bounds. It is done heuristically by performing one of the following substitutions:

\[ x_j = l_j + x'_j; \ x_j = v_j - x'_j; \ x_j = \tilde{l}_j y_i + x'_j; \ w_j = \tilde{v}_j y_k - x'_j \]
Bound substitution

Consider the mixed-integer set
\[ X^{MI} = \{ (y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : ay + gx \leq b, \ y \leq u, lx \leq v \}. \]

The MIP formulation can also include some additional variable bounds on the continuous variables.

**Bound substitution** consists of replacing some continuous variables by their respective simple/variable bounds. It is done heuristically by performing one of the following substitutions:

\[ x_j = l_j + x'_j; \ x_j = v_j - x'_j; \ x_j = \tilde{l}_j y_i + x'_j; \ w_j = \tilde{v}_j y_k - x'_j \]

Let \((\bar{y}, \bar{x})\) be the current fractional solution. The bound with smallest slack is selected for substitution. That is, let

\[ \mu = \min \{ \bar{x}_j - l_j, \ v_j - \bar{x}_j, \ \bar{x}_j - \tilde{l}_j \bar{y}_i, \ \tilde{v}_j \bar{y}_k - \bar{x}_j \}. \]
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Let:

\[
x_j = \begin{cases} 
  l_j + x'_j & \text{if } \mu = x_j - l_j \\
  v_j - x'_j & \text{if } \mu = v_j - \bar{x}_j \\
  \tilde{l}_j y_i + x'_j & \text{if } \mu = \bar{x}_j - \tilde{l}_j \\
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\end{cases}
\]
Preprocessing

Let

\[
\sum_{i \in I} a_i' y_i + \sum_{j \in P} g_j' x_j' \leq b',
\]

with \(0 \leq y_i \leq u_i \ \forall j \in I\) and \(x_j' \geq 0 \ \forall j \in P\), be the mixed-integer inequality after bound substitution.
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- All the continuous variables with positive coefficients can be discarded (Atamturk, 2000).
- All the continuous variables with negative coefficients are aggregated into the same variable \( s \):

\[ s = - \sum_{j \in P^-} g'_j x'_j, \]

where \( P^- = \{ j \in P : g'_j < 0 \} \).
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$$y_j = \begin{cases} u_j - y_j' & \text{if } a_j' < 0 \\ y_j' & \text{otherwise} \end{cases}$$
Convert all the coefficients into integers

- The integer knapsack problems of \textit{MKNAP} are solved by the dynamic programming algorithm of Pisinger (2001).
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- Dynamic programming is fast, but there is a price to pay: it requires that all the knapsack coefficients are integers.

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- We adopt a brute-force approach: enumerate all the \( q \in \mathbb{N} \) in the interval \( \left[ 1, 10^4 \right] \), stopping when \( q_{b''} - \lfloor q_{b''} \rfloor \leq \varepsilon \) and \( q_{a'' j} - \lfloor q_{a'' j} \rfloor \leq \varepsilon \) for each \( j \in I \). In our experiments we set \( \varepsilon = 10^{-5} \).

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- Computing a lifting coefficient amounts to solve a knapsack problem with a single continuous variable. The problem can be solved by splitting into two integer knapsack problems.
Computational results

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- The test bed consists of all the MIPLIB 2003 mixed-integer instances and of the “Mittleman” instances $bc1, bienst1, bienst2, binkar10_1, dano3-4, dano3-5$. We set a limit of 300 CPU secs for the time spent in separation.
Computational results (cont.)

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For simplicity of comparison, separation routines run on the original (i.e. not preprocessed) instances.
## Computational results

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The overall effect

Some preliminary tests on non-trivial instances (Cplex 11.1)

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Some considerations

- Computational experience shows that exact separation for $\text{conv}(X^M^K)$ is effective in tightening MIP formulations.
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- Exact separation not applicable to large and dense rows.
We focus on mixed knapsack inequalities (Marchand and Wolsey, 2002), which can be described by the following procedure.

Given:

\[ X^{BMK} = \{(y, s) \in \mathbb{B}_+^n \times \mathbb{R}_+ : ay - s \leq b, \ y \leq u\} \]
Back to the template paradigm: Mixed Knapsack Inequalities

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- ii) Find a valid inequality \( \alpha y \leq \beta \) for the resulting binary knapsack polytope;

\[ X_{\bar{s}}^{BMK} = \{y \in \mathbb{B}_+^n : ay \leq b + \bar{s}, \ y \leq u\} \]
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\[ X^{BMK}_{\bar{s}} = \{y \in B^n_+ : ay \leq b + \bar{s}, y \leq u\} \]

iii) lift the \( s \) to get a valid inequality for \( X_{BMK} \) of the form \( \alpha y - \gamma s \leq \beta \).
Mixed Knapsack Inequalities: lifting the $s$

Let

\[ \eta(s) = \max \alpha y \] (5)

\[ ay \leq b + s \] (6)

\[ y \in \{0, 1\}^n \] (7)
Mixed Knapsack Inequalities: lifting the $s$

Let

$$\eta(s) = \max \alpha y$$ \hspace{1cm} (5)

$$\alpha y \leq b + s$$ \hspace{1cm} (6)

$$y \in \{0, 1\}^n$$ \hspace{1cm} (7)

**Proposition**

*The inequality*

$$\alpha y \leq \beta + \gamma s$$

*is valid for $\text{conv}(X^{BMK})$ if $\eta(s) \leq \beta + \gamma s$ for each $s \in \mathbb{R}_+$.***
A geometrical interpretation
Mixed Knapsack Inequalities: lifting the s (cont.)

A geometrical interpretation

- \( \eta(s) \) is a step function
A geometrical interpretation

- $\eta(s)$ is a step function

- The line $\beta + \gamma s$ is a “valid” rhs if it defines an upper bound on the $\eta(s)$, for each $s \in \mathbb{R}_+$. 
Mixed Knapsack Inequalities: lifting the $s$ (cont.)

Step 0: Initialize $\gamma$.

Step 1: Solve the problem:

$$\zeta = \max \alpha y - \gamma s$$

subject to:

$$\alpha y \leq b + s y \in \{0, 1\}$$

$N s \geq 0$

Step 2: If $\zeta \leq \beta$ then the inequality $\alpha y \leq \beta + \gamma s$ is valid for $\text{conv}(X_{BMK})$. STOP.

Step 3: Increase $\gamma$ and Go to Step 1.
Mixed Knapsack Inequalities: lifting the $s$ (cont.)

The lifting algorithm
Mixed Knapsack Inequalities: lifting the s (cont.)

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$$
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$$

$$
\alpha y \leq b + s
$$

$$
y \in \{0, 1\}^N
$$

$$
s \geq 0
$$

If $\zeta \leq \beta$ then the inequality $\alpha y \leq \beta + \gamma s$ is valid for $\text{conv}(X_{BMK})$. STOP.

Step 3 Increase $\gamma$ and Go to Step 1.
Mixed Knapsack Inequalities: lifting the $s$ (cont.)

The lifting algorithm

**Step 0** Inizialize $\gamma$.

**Step 1** Solve the problem:

$$\zeta = \max \alpha y - \gamma s$$
$$\alpha y \leq b + s$$
$$y \in \{0, 1\}^N$$
$$s \geq 0$$

**Step 2** If $\zeta \leq \beta$ then the inequality $\alpha y \leq \beta + \gamma s$ is valid for $\text{conv}(X^{BMK})$. **STOP.**
Mixed Knapsack Inequalities: lifting the s (cont.)

The lifting algorithm

**Step 0** Initalize $\gamma$.

**Step 1** Solve the problem:

$$\zeta = \max \alpha y - \gamma s$$

$$\alpha y \leq b + s$$

$$y \in \{0, 1\}^N$$

$$s \geq 0$$

**Step 2** If $\zeta \leq \beta$ then the inequality $\alpha y \leq \beta + \gamma s$ is valid for $\text{conv}(X_{BMK}^b)$. **STOP**.

**Step 3** Increase $\gamma$ and **Go to** Step 1.
A numerical example

Consider the set $X_{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$.

Let $X_{BMK1} = \{x \in X_{BMK}: s = 1\}$.

The inequality $y_1 + y_2 + y_3 + y_4 \leq 1$ is valid for $\text{conv}(X_{BMK1})$. 
Mixed Knapsack Inequalities: lifting the s (cont.)

A numerical example

- Consider the set $X^{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$. 
Mixed Knapsack Inequalities: lifting the $s$ (cont.)

A numerical example

- Consider the set $X^{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$.
- Let $X_1^{BMK}\{x \in X^{BMK} : s = 1\}$. 
Mixed Knapsack Inequalities: lifting the $s$ (cont.)
A numerical example

- Consider the set $X^{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$.
- Let $X_1^{BMK} \{x \in X^{BMK} : s = 1\}$.
- The inequality $y_1 + y_2 + y_3 + y_4 \leq 1$ is valid for $conv(X_1^{BMK})$. 
Mixed Knapsack Inequalities: lifting the $s$ (cont.)

A numerical example

- Consider the set $X^{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$.
- Let $X_1^{BMK} = \{x \in X^{BMK} : s = 1\}$.
- The inequality $y_1 + y_2 + y_3 + y_4 \leq 1$ is valid for $\text{conv}(X_1^{BMK})$.
- $\eta(s)$ step function.
Mixed Knapsack Inequalities: lifting the $s$ (cont.)

A numerical example

- Consider the set $X^{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$.
- Let $X_1^{BMK} = \{x \in X^{BMK} : s = 1\}$.
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- Initialization: $\gamma = 3/15$. 
Mixed Knapsack Inequalities: lifting the $s$ (cont.)

A numerical example

- Consider the set $X^{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$.
- Let $X_1^{BMK} = \{x \in X^{BMK} : s = 1\}$.
- The inequality $y_1 + y_2 + y_3 + y_4 \leq 1$ is valid for $\text{conv}(X_1^{BMK})$.
- **Iteration 1**: update $\gamma = 1; y_1 + y_2 + y_3 + y_4 - s \leq 1$ is valid.
## Computational results for Mixed Knapsack Inequalities

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Research directions

- More efficient ways of solving the exact separation LP
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- How to generate new rows by constraint aggregation?
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Research directions

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- How to generate new rows by constraint aggregation?
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- How to select MIP substructures to ensure that exact separation leads to violated cuts?