Two-Period Relaxations on Big-Bucket Production Planning Problems

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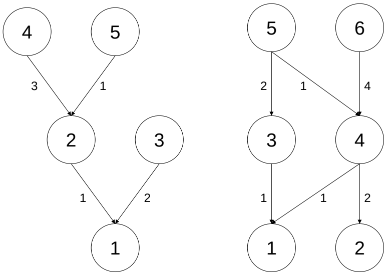
Problem Description

- Multiple items and levels (BOM structure)
  - Assembly (a) or general (b) structures



(a) (b)

- Demands
- Big-bucket capacities (items share resources)
- Extensions possible, e.g. overtime and backlogging
- Production plan minimizing total cost to be determined

Basic Formulation

$$\min \sum_{t=1}^{NT}\sum_{i=1}^{NI} f_t^i y_t^i + \sum_{t=1}^{NT}\sum_{i=1}^{NI} h_t^i s_t^i \quad (1)$$

$$\text{s.t. } x_t^i + s_{t-1}^i - s_t^i = d_t^i \qquad t \in [1, NT], i \in endp \quad (2)$$

$$x_t^i + s_{t-1}^i - s_t^i = \sum_{j \in \delta(i)} r^{ij} x_t^j \qquad t \in [1, NT], i \notin endp \quad (3)$$

$$\sum_{i=1}^{NI} (a_k^i x_t^i + ST_k^i y_t^i) \leq C_t^k \qquad t \in [1, NT], k \in [1, NK] \quad (4)$$

$$x_t^i \leq M_t^i y_t^i \qquad t \in [1, NT], i \in [1, NI] \quad (5)$$

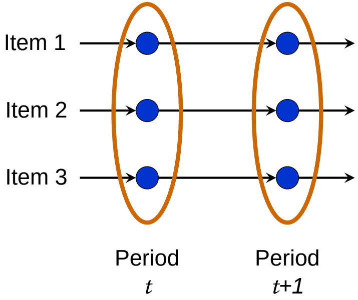
$$y \in \{0, 1\}^{NTxNI} \quad (6)$$

$$x \geq 0 \quad (7)$$

$$s \geq 0 \quad (8)$$

What do we know?

- Many of the test problems remain challenging
- We do not have an adequate approximation of the convex hull of the **multi-item, single-machine, single-level capacitated problems**!



- Generalizes the "bottleneck flow" model of Atamtürk and Muñoz [2003]

The Model to Study

$$x^i_{t'} \leq \widetilde{M}^i_{t'} y^i_{t'} \qquad i = [1, ..., NI], t' = 1, 2 \qquad (9)$$

$$x^i_{t'} \leq \widetilde{d}^i_{t'} y^i_{t'} + s^i \qquad i = [1, ..., NI], t' = 1, 2 \qquad (10)$$

$$x^i_1 + x^i_2 \leq \widetilde{d}^i_1 y^i_1 + \widetilde{d}^i_2 y^i_2 + s^i \qquad i = [1, ..., NI] \qquad (11)$$

$$x^i_1 + x^i_2 \leq \widetilde{d}^i_1 + s^i \qquad i = [1, ..., NI] \qquad (12)$$

$$\sum_{i=1}^{NI} (a^i x^i_{t'} + ST^i y^i_{t'}) \leq \widetilde{C}_{t'} \qquad t' = 1, 2 \qquad (13)$$

$$x, s \geq 0, y \in \{0, 1\}^{2 \times NI} \qquad (14)$$

- Let $X^{2PL} = \{(x, y, s) | (9) - (14)\}$

Separation Over the 2-Period Convex Hull: Details

From the LPR of the original problem, we obtain $(\bar{x}, \bar{y}, \bar{s})$

$$\min\ z = \sum_i [(\Delta_s^-)^i + \sum_{t'=1}^{2} (\Delta_x^+)_{t'}^i + (\Delta_x^-)_{t'}^i + (\Delta_y^+)_{t'}^i + (\Delta_y^-)_{t'}^i]$$

$$\text{s.t. } \bar{x}_{t'}^i = \sum_k \lambda_k (x_k)_{t'}^i + (\Delta_x^+)_{t'}^i - (\Delta_x^-)_{t'}^i \quad \forall i, t' = 1,2 \quad (\alpha_{t'}^i)$$

$$\bar{y}_{t'}^i = \sum_k \lambda_k (y_k)_{t'}^i + (\Delta_y^+)_{t'}^i - (\Delta_y^-)_{t'}^i \quad \forall i, t' = 1,2 \quad (\beta_{t'}^i)$$

$$\bar{s}^i \geq \sum_k \lambda_k (s_k)^i - (\Delta_s^-)^i \quad \forall i \quad (\gamma^i)$$

$$\sum_k \lambda_k \leq 1 \quad (\eta)$$

$$\lambda_k \geq 0,\ \Delta \geq 0$$

Separation Over the 2-Period Convex Hull: Details (cont'd)

The dual of the distance problem:

$$\max \sum_{i=1}^{NI} \sum_{t'=1}^{2} (\bar{x}^i_{t'} \alpha^i_{t'} + \bar{y}^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} \bar{s_k}^i \gamma^i + \eta \qquad (15)$$

$$\text{s.t.} \quad \sum_{i=1}^{NI} \sum_{t'=1}^{2} ((x_k)^i_{t'} \alpha^i_{t'} + (y_k)^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} (s_k)^i \gamma^i + \eta \leq 0 \qquad \forall k \qquad (16)$$

$$-1 \leq \alpha^i_{t'} \leq 1 \qquad \forall i, t' \qquad (17)$$

$$-1 \leq \beta^i_{t'} \leq 1 \qquad \forall i, t' \qquad (18)$$

$$-1 \leq \gamma^i \leq 0 \qquad \forall i \qquad (19)$$

$$\eta \leq 0 \qquad (20)$$

Separation Over the 2-Period Convex Hull: Details (cont'd)

Theorem

Let $z > 0$ for $(\bar{x}, \bar{y}, \bar{s})$, and $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$ optimal dual values. Then,

$$\sum_{i=1}^{NI} \sum_{t'=1}^{2} (\bar{\alpha}^i_{t'} x^i_{t'} + \bar{\beta}^i_{t'} y^i_{t'}) + \sum_i \bar{\gamma}^i s^i + \bar{\eta} \leq 0 \qquad (21)$$

is a valid inequality for $conv(X^{2PL})$ that cuts off $(\bar{x}, \bar{y}, \bar{s})$.

Proof.

Validity: Using (16), $\bar{\gamma} \leq 0$ and $\lambda \geq 0$.
Violation for $(\bar{x}, \bar{y}, \bar{s})$: Using (15) □

Separation Over the 2-Period Convex Hull: Extreme Points

- How to generate (x_k, y_k, s_k)?
  - Using column generation
  - Solve the minimum reduced cost problem using the optimal dual values $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$:

$$\max_{x,y,s} \; z_P = \sum_i \sum_{t'=1}^{2} (\bar{\alpha}^i_{t'} x^i_{t'} + \bar{\beta}^i_{t'} y^i_{t'}) + \sum_i \bar{\gamma}^i s^i + \bar{\eta}$$
$$\text{s.t. } (x, y, s) \in X^{2PL}$$

  - If $z_P > 0$, then the solution is an extreme point of X^{2PL}; otherwise, generating extreme points is done

Separation Algorithm

repeat

Solve the distance problem for $conv(X^{2PL})$

if z = 0 **then** break

else Solve column generation problem

if $z_P \leq 0$ **then** break

else Add new extreme point

until z = 0 **or** $z_P \leq 0$

if z=0 **then** $(\bar{x}, \bar{y}, \bar{s}) \in conv(X^{2PL})$

else Add the violated cut (21)

Note: The same is valid for $\mathcal{L}_\infty$

Using Euclidean Distance ($\mathcal{L}_2$)

We can also apply the same framework with Euclidean distance

$$\min_{\Delta,\lambda} \; z = \sum_i \left[[(\Delta_s)^i]^2 + \sum_{t'=1}^{2} \left([(\Delta_x)^i_{t'}]^2 + [(\Delta_y)^i_{t'}]^2 \right) \right]$$

$$\begin{aligned}
\text{s.t.}\; \bar{x}^i_{t'} &= \sum_k \lambda_k (x_k)^i_{t'} + (\Delta_x)^i_{t'} && \forall i, t' = 1,2 && (\alpha^i_{t'}) \\
\bar{y}^i_{t'} &= \sum_k \lambda_k (y_k)^i_{t'} + (\Delta_y)^i_{t'} && \forall i, t' = 1,2 && (\beta^i_{t'}) \\
\bar{s}^i &\geq \sum_k \lambda_k (s_k)^i - (\Delta_s)^i && \forall i && (\gamma^i) \\
\sum_k \lambda_k &\leq 1 && && (\eta) \\
\lambda_k \geq 0,\ &\Delta_s \geq 0,\ \Delta_x, \Delta_y \text{ free}
\end{aligned}$$

Using Euclidean Distance: Dual

$$\max_{\Delta,\alpha,\beta,\gamma} z_D = -\sum_i \left[[(\Delta_s)^i]^2 + \sum_{t'=1}^{2} [(\Delta_x)^i_{t'}]^2 + [(\Delta_y)^i_{t'}]^2 \right]$$

$$- \left(\sum_{i=1}^{NI} \sum_{t'=1}^{2} (\bar{x}^i_{t'} \alpha^i_{t'} + \bar{y}^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} \bar{s}^i \gamma^i + \eta \right)$$

$$\text{s.t.} \quad \sum_{i=1}^{NI} \sum_{t'=1}^{2} ((x_k)^i_{t'} \alpha^i_{t'} + (y_k)^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} (s_k)^i \gamma^i + \eta \geq 0 \quad \forall k$$

$$\alpha^i_{t'} = -2(\Delta_x)^i_{t'} \quad \forall i, t'$$

$$\beta^i_{t'} = -2(\Delta_y)^i_{t'} \quad \forall i, t'$$

$$-\gamma^i \geq -2(\Delta_s)^i \quad \forall i$$

$$\gamma \geq 0, \ \eta \geq 0, \ \Delta_s \geq 0, \ \alpha, \beta, \Delta_x, \Delta_y \ \text{free}$$

Using Euclidean Distance: Theory

Theorem

Let $z > 0$ for $(\bar{x}, \bar{y}, \bar{s})$, with optimal primal values $(\bar{\Delta}_x, \bar{\Delta}_y, \bar{\Delta}_s, \bar{\lambda})$, and $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$ be the associated optimal dual values. Then,

$$\sum_i \sum_{t'=1}^{2} (\bar{\alpha}^i_{t'} x^i_{t'} + \bar{\beta}^i_{t'} y^i_{t'}) + \sum_i \bar{\gamma}^i s^i + \bar{\eta} \geq 0 \qquad (22)$$

is a valid inequality for $conv(X^{2PL})$ that cuts off $(\bar{x}, \bar{y}, \bar{s})$.

Proof.

Using a similar approach to the previous proof and also using the strong duality theorem for QP. □

Defining 2-Period Subproblems

- **Question 1:** On which two periods to run the separation?
  - We can look at all the 2-period problems ($NT - 1$ of them)
- **Question 2:** Which period's stock is represented by s^i?
  - Let $\phi(i) \in [t+1, .., NT]$ be the horizon parameter for each i
  - Obvious choice: $t+1$, i.e., $s^i = s^i_{t+1}$
  - Then, parameters are defined as follows ($\forall i,\ t' = 1, 2$):
    - $\widetilde{M}^i_{t'} = M^i_{t+t'-1}$, $\widetilde{C}^i_{t'} = C^i_{t+t'-1}$
    - $\widetilde{d}^i_{t'} = d^i_{t+t'-1,\ t+1}$, i.e., $\widetilde{d}^i_1 = d^i_{12}$ and $\widetilde{d}^i_2 = d^i_2$.
- **Observation 1:** If a number of periods following $t+1$ have no setups, their demands should be incorporated
- **Observation 2:** If a setup occurs after $t+1$, (ℓ, S) inequalities will be weakened if that period's demand is in

2-Period Convex Hull Closure Framework

- Following Miller, Nemhauser, Savelsbergh (2000)

$$\phi(i) = \max\{t' | t' \geq t+1, \sum_{t''=t+1}^{t'} y^i_{t''} \leq y^i_{t+1} + \Theta\}$$

  where $\Theta \in (0,1]$ is a random number
- Let X_t^{2PL} be $X_t^{2PL}(\phi(1), \phi(2), ..., \phi(NI))$

  Solve LPR of the original problem
  $\rightarrow (\bar{x}, \bar{y}, \bar{s})$
  for t=1 **to** NT-1
  Define X_t^{2PL}
  Apply 2-period convex hull separation algorithm

Computational Results: 2-Period Problems

- *2PCLS* instances: 20 problems with two periods only and with two to six items
  - **cdd** might provide the full description of the convex hull
    - Generate all the extreme points and rays of the LPR
    - Eliminate all the fractional extreme points
    - Using these integral extreme points, generate all the facets of the integral polyhedron
  - The more items share a resource, the more the structure tends to resemble that of an uncapacitated problem
- The separation algorithm implemented in Mosel (Mosel version 2.0.0, Xpress 2007 package)

Computational Results: 2-Period Problems (cont'd)

Inst.	NI	XLP	IP	# cuts ($\mathcal{L}_1$)	# cuts ($\mathcal{L}_\infty$)	# cuts ($\mathcal{L}_2$)
2pcls01	3	17.033	25	11	8	19
2pcls02	3	12.6253	19	13	7	7
2pcls03	3	76.5345	104	5	3	1
2pcls04	2	14.7674	19	4	2	1
2pcls05	3	38.39	52	8	6	4
2pcls06	3	117.375	173	5	6	5
2pcls07	2	36.5	43	2	1	1
2pcls08	2	21.45	26	7	2	2
2pcls09	2	129	153	2	3	3
2pcls10	3	17.6539	24	1	3	1

Computational Results: 2-Period Problems (cont'd)

Inst.	NI	XLP	IP	# cuts ($\mathcal{L}_1$)	# cuts ($\mathcal{L}_\infty$)	# cuts ($\mathcal{L}_2$)
2pcls11	3	71.7209	102	4	1	1
2pcls12	3	46.68	69	4	1	2
2pcls13	4	85.6256	113	7	7	9
2pcls14	4	70.2961	81	6	8	5
2pcls15	4	54.1848	74	6	3	1
2pcls16	4	34.0844	39	6	7	4
2pcls17	5	164.858	211	39	19	14
2pcls18	5	57.0825	97	34	10	6
2pcls19	6	115.131	150	11	6	1
2pcls20	6	59.2412	89	34	11	5

Computational Results: Multi-Period Problems

tr6-15 detailed results (30 iterations):

	$\mathcal{L}_1$		$\mathcal{L}_2$		
	lim=100 $\phi(i)$	lim=150 $\phi(i)$	lim=100 $t+1$	lim=150 $t+1$	lim=150 $\phi(i)$
2PL	37,234.6	37,298.8	37,306.7	37,306.8	37,331.2

Trigeiro instances results with Euclidean approach:

	tr6-15	tr6-30	tr12-15	tr12-30
XLP	37,201	60,946	73,848	130,177
2PL	37,344	61,090	73,932	130,215
OPT	37,721	61,746	74,634	130,596

Computational Results: Multi-Period Problems (cont'd)

TDS instances results with $\mathcal{L}_\infty$ approach:

	AK501131	AK501132	AK501141	AK502131	AK502132
LB (l,S)	96,968	101,699	134,805	93,369	96,312
LB (2PL)	114,788	108,846	150,954	107,477	110,542
Best Sol.	123,366	123,473	170,897	115,819	118,319
Gap was	27.22%	21.41%	26.77%	24.04%	22.85%
Gap now	7.47%	13.44%	13.21%	7.76%	7.04%

	BK511131	BK511141	BK521131	BK521142	BK522142
LB (l,S)	92,602	125,307	92,350	124,988	119,559
LB (2PL)	101,565	134,997	106,302	139,090	131,552
Best Sol.	120,303	172,762	118,217	154,258	148,471
Gap was	29.91%	37.87%	28.01%	23.42%	24.18%
Gap now	18.45%	27.97%	11.21%	10.91%	12.86%

Conclusions

- Study of a stronger relaxation
  - A new framework independent from defining families of valid inequalities or reformulations a priori, although expected output is to define new valid inequalities using the results from the framework
  - To our knowledge, this is an original approach in production planning literature
- Different norms useful to generate cuts and improve lower bounds significantly
  - Euclidean and $\mathcal{L}_\infty$ approaches computationally much more efficient than Manhattan approach
    - Observed both on the efficiency of cuts and on the number of extreme points generated in column generation before termination

Future Directions

- Immediate directions
  - Achieving more computational results on realistic size problems
  - Applying lifting on the generated inequalities
    - Possibility to obtain facets instead of faces
- Near-future directions
  - Careful analysis of the inequalities generated by the framework and the facets from **cdd**
    - Significant potential to identify new families of valid inequalities
- Possibilities for extending results to other MIP techniques
  - *Example:* Disjunctive cuts
    - Normalization or Manhattan distance used by researchers
    - Euclidean has potential to provide more efficient cuts
- Extension to other MIP problems possible
  - "Local Cuts" of Cook, Espinoza and Chvátal [2006]