MIP Lifting Techniques for Mixed Integer Nonlinear Programs

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Structure of the Talk

Introduction & Literature Review

Motivation; Lifting; Literature Review; . . .

Example 1: Single Constraint Nonlinear Sets

Applications; Linearizations of Bilinear Sets; Superadditive Lifting; Lifted Covers; Lifted Cliques; High Rank Certificate

Example 2: Nonlinear Disjunctive Sets

General Procedure; Two-terms disjunction; General Result; Three-terms disjunction;

Some Theoretical Results

Conjugate & Bi-conjugate; Sequence Independence; Superadditive Lifting;

Conclusion

Part I:

Introduction and Literature Review

A Motivation in Integer Programming

Observation 1: Solving linear integer programs efficiently is often related to the ability of creating tight (LP) relaxations.

Observation 2: In linear integer programming, tight relaxations can be obtained

1. By reformulating the problem in a higher dimensional space
2. By adding cutting planes in the space of original variables

Observation 3: In linear unstructured integer programming, the above two tightenings can be obtained through

1. Disjunctive Arguments
2. Lifting

A Motivation in Integer Programming

Observation 1: Solving nonlinear integer programs efficiently is often related to the ability of creating tight convex relaxations.

Observation 2: For MINLP, tight convex relaxations can be obtained

1. By reformulating the problem in a higher dimensional space
2. By adding "convex" cuts in the space of original variables

Observation 3: For MINLP,

1. Disjunctive Arguments have been used to obtained tighter formulation
2. Lifting has not been used!

What is MIP Lifting?

For a set

$$S = \{x \in \mathbb{Z}^m \times \mathbb{R}^{n-m} \,|\, Ax \leq b\}$$

And a restriction of this set

$$S(T, u) = \{x \in S \,|\, x_j = u_j \forall j \in T\}$$

Lifting is the process by which a valid inequality of $S(T, u)$ of the form

$$\sum_{j \in N \setminus T} \alpha_j x_j \leq \delta$$

is converted into a valid inequality of S of the form

$$\sum_{j \in N} \alpha_j x_j \leq \delta + \sum_{j \in T} \alpha_j u_j$$

Some Literature on MIP lifting

Lifting has a relatively long history:

1. Integer variables: $\sim$ 1970's
   (a) Sequence Dependent: Padberg; Wolsey; Balas & Zemel; . . .
   (b) Sequence Independent: Wolsey; Gu, Nemhauser & Savelsbergh; Atamtürk; . . .

2. Continuous variables: $\sim$ 2000's
   (a) Sequence Dependent: Marchand and Wolsey; R., de Farias & Nemhauser
   (b) Sequence Independent: R., de Farias & Nemhauser; R.

What is Hard about MINLP Lifting?

1. In MIP lifting, we just need to guess the value of a coefficient:

   ≻ Consider the set $S = \{x \in \{0,1\}^4 \,|\, 20x_1 + 15x_2 + 12x_3 + 10x_4 \leq 22\}$

   ≻ The cover inequality $x_2 + x_3 + x_4 \leq 2$ is valid for $S(x_1, 0)$.

   ≻ To obtain the lifted inequality, we just need to guess the coefficient in front of x_1, $2x_1 + x_2 + x_3 + x_4 \leq 2$

2. In MINLP Lifting, we also need to guess the shape of the function:

   ≻ Consider the set $S = \{x \in [0,1]^3 \,|\, 2xy + \frac{1}{2}z \geq 1\}$.

   ≻ The convex hull of feasible solutions is given by $\sqrt{2xy} + (1 - \sqrt{\frac{1}{2}})z \geq 1$

Overview & Goal of Our Work

1. Our goal is to extend lifting techniques to nonlinear mixed integer programs

2. The main difference with existing techniques for constraints generation in MINLP is that cuts are generated in the original space of variables

3. We will proceed in two steps

   (a) Obtaining valid linear cuts for MINLP sets

   (b) Obtaining valid nonlinear cuts for MINLP sets

4. Some isolated use of lifting for MINLP: de Farias, Lin & Vandenbussche

Some Nice Features of MINLP Lifting

MINLP lifting is capable of producing

1. Strong nonlinear (convex) inequalities for NLP and MINLP

2. Linear inequalities that are difficult to obtain through linearizations

Part II:

Example 1: Single Constraint Nonlinear Sets

Goal of Part II

Illustrate that:

Although it is sometimes possible to study nonlinear sets through linearizations, stronger results might be obtained by focusing directly on the nonlinear formulation

Part II.1:

A General Single Constraint Mixed Integer Nonlinear Knapsack Set

Mixed Integer Nonlinear Knapsack

Problem Definition: We consider the set

$$S = conv\{(x,y) \in \{0,1\}^n \times [0,1]^n \mid \sum_{j=1}^{n} f_j(x_j,y_j) \leq d\}$$

where

$$f_j : \{0,1\} \times [0,1] \to \mathbb{R}$$
$$d \in \mathbb{R}$$

Remark: S is generalization of

1. Linear Mixed Integer Knapsack Polytopes:
   $\to$ take $f_j(x_j,y_j) = a_j x_j + b_j y_j$ for $a_j, b_j \in \mathbb{R}$.
2. Single Node Fixed Charge Flow Models:
   $\to$ take $f_j(x_j,y_j) = a_j y_j$ if $y_j \leq x_j$ and $f_j(x_j,y_j) = \infty$ otherwise.
3. Bilinear Mixed Integer Knapsack Sets:
   $\to$ take $f_j(x_j,y_j) = a_j x_j y_j$ for $a_j, b_j \in \mathbb{R}$.

Mixed Integer Nonlinear Knapsack

Notation:

Given the mixed integer nonlinear knapsack set S, we let $PS = conv\{S\}$ be the convex hull of S.

Remarks:

1. PS is usually not a polyhedron.

2. We want to derive "strong" valid inequalities for PS.

3. We will use lifting to obtain these inequalities.

4. Most of the results we describe next also apply when f_j is a function of more than 2 variables.

Generating Valid Inequalities for PS

Definition: Given S and $T \subseteq N$, we define

$$S(T) = \{(x, y) \in S \,|\, x_j = y_j = 0, \forall j \in T\}$$

and define $PS(T) = conv(S(T))$.

Definition: Let $i \in N$, given a linear inequality

$$\sum_{j=1}^{i} \alpha_j x_j + \sum_{j=1}^{i} \beta_j y_j \leq \delta (*)$$

that is valid for $PS(\{i+1, \ldots, n\})$, we define the lifting function associated with (*) to be

$$\phi^i(a) = \delta - \quad \max \quad \{\sum_{j=1}^{i} \alpha_j x_j + \sum_{j=1}^{i} \beta_j y_j\}$$

$$s.t. \quad \sum_{j=1}^{i} f_j(x_j, y_j) \leq d - \sum_{j=i+1}^{n} f_j(0, 0)$$

A General Lifting Result for PS

Proposition: Assume that

$$\alpha_1 x_1 + \beta_1 y_1 \leq \delta$$

is valid for $PS(\{2, \ldots, n\})$. For $i = 2, \ldots, n$, also assume that the lifting coefficients α_i and β_i are chosen such that

$$\alpha_i + \beta_i \theta \leq \phi^{i-1}(f_i(1, \theta) - f_i(0, 0)) \qquad \forall 0 \leq \theta \leq 1$$
$$\beta_i \theta \leq \phi^{i-1}(f_i(0, \theta) - f_i(0, 0)) \qquad \forall 0 < \theta \leq 1$$

Then the inequality

$$\sum_{j=1}^{i} \alpha_j x_j + \sum_{j=1}^{i} \beta_j y_j \leq \delta$$

is valid for $PS(\{i+1, \ldots, n\})$

Advantages and Limitations of the Lifting Scheme

Advantages:

1. Systematic and constructive method to generate cuts.

2. If performed "exactly," this scheme generates "strong" inequalities.

Limitations:

1. The computation of each single function ϕ^i might be difficult and/or computationally prohibitive.

2. All functions ϕ^i need to be computed.

3. All inequalities generated are linear.

A Superadditive Lifting Result for PS

Proposition: Assume that

$$\alpha_1 x_1 + \beta_1 y_1 \leq \delta$$

is valid for $PS(\{2,\ldots,n\})$. Also assume that $\phi^1(a)$ is superadditive, i.e. $\phi^1(a)+\phi^1(b) \leq \phi^1(a+b)$, $\forall a,b \in \mathbb{R}$. Then if the lifting coefficient α_i and β_i are chosen such that

$$\begin{aligned} \alpha_i + \beta_i\theta &\leq \phi^1(f_i(1,\theta) - f_i(0,0)) \qquad \forall 0 \leq \theta \leq 1 \\ \beta_i\theta &\leq \phi^1(f_i(0,\theta) - f_i(0,0)) \qquad \forall 0 < \theta \leq 1 \end{aligned}$$

Then the inequality

$$\sum_{j=1}^{i} \alpha_j x_j + \sum_{j=1}^{i} \beta_j y_j \leq \delta$$

is valid for $PS(\{i+1,\ldots,n\})$

A Superadditive Lifting Result for PS

Proof: [Sketch]

◇ Note first that $\Phi^i(a) \leq \Phi^{i-1}(a)$, $\forall i = 2, \ldots, n-1$

◇ The rest of the proof is by induction.

◇ Use dynamic programming to prove that

$$\Phi^i(a) = min \begin{cases} \Phi^{i-1}(a) \\ \inf_{0<\theta\leq 1}\{\Phi^{i-1}(a + f_i(0,\theta^*) - f_i(0,0) - \beta_i\theta^*)\} \\ \inf_{0\leq\theta\leq 1}\{\Phi^{i-1}(a + f_i(1,\theta^*) - f_i(0,0) - \alpha_i - \beta_i\theta^*)\} \end{cases}$$

◇ Use inductively proved superadditivity of Φ^{i-1} to argue

$$\Phi^i(a) \geq min \begin{cases} \Phi^{i-1}(a) \\ \Phi^{i-1}(a) + \inf_{0<\theta\leq 1}\{\Phi^{i-1}(f_i(0,\theta^*) - f_i(0,0) - \beta_i\theta^*)\} \\ \Phi^{i-1}(a) + \inf_{0\leq\theta\leq 1}\{\Phi^{i-1}(f_i(1,\theta^*) - f_i(0,0) - \alpha_i - \beta_i\theta^*)\} \end{cases}$$

◇ Note that the conditions dictating the choice of α_i, β_i imply

$$\Phi^i(a) \geq \Phi^{i-1}(a)$$

◇ Conclude that $\Phi^i(a) = \Phi^{i-1}(a)$

A Superadditive Lifting Result for PS

Comments

1. This theorem generalizes sequence independent lifting for particular forms of nonlinear mixed integer programs

2. It can be used even if the f_j have different expressions

3. Provided that the lifting coefficients α_j and β_j can be derived quickly from ϕ, it provides an efficient way to generate constraints for MINLP

Part II.2:

A Single Constraint Mixed Integer Bilinear Knapsack Set

Application: Bilinear Mixed Integer Knapsack Problem (BMIKP)

We now apply the previous result on the set

$$T = conv\{(x,y) \in \{0,1\}^n \times [0,1]^n \mid \sum_{j \in N} a_j x_j y_j \le d\}$$

where

1. $a_j > 0$, $\forall j \in N$
2. $d > 0$

This set can appear as a relaxation of

1. Outsourcing problems
2. Environment problems
3. ...

BMIKP: Comments

Remarks

1. $PT = conv(T)$ is a polytope

2. We will investigate the facets of PT
   We proceed in two steps

   (a) Derive the inequalities of the convex hull of PT', the continuous relaxation of PT.

   (b) Derive some facets of PT using the superadditive theory.

The Convex Hull of PT' is a Polyhedron

Proposition Let PT' be defined as

$$PT' = conv\{(x, y) \in [0,1]^n \times [0,1]^n \mid \sum_{j=1}^{n} a_j x_j y_j \leq d\}$$

Then

1. PT' is a polytope

2. $PT' = \{(x, y) \in [0,1]^n \times [0,1]^n | \sum_{j \in T} a_j x_j + \sum_{j \in T} a_j y_j \leq d + \sum_{j \in T} a_j, \forall T \subseteq N\}$

Obtaining Facets of PT using Superadditive Lifting

Definition: Given $C \subseteq N$, we say that C is a cover for PT if

$$\sum_{j \in C} a_j = d + \mu$$

for some $\mu > 0$.

Definition: We say that C is a nontrivial cover if $\max_{i \in C} a_i > \mu$.

Proposition: Let $C \subseteq N$ and T_0, T_1 be a partition of $N \backslash C$. Assume that C is a nontrivial cover in $PT(N \backslash C, \emptyset, \emptyset, N \backslash C)$, then the inequality

$$\sum_{j \in C} \min\{a_j, \mu\} x_j + \sum_{j \in C} a_j y_j \leq \sum_{j \in C} \min\{a_j, \mu\} + \sum_{j \in C} a_j - \mu$$

is a facet of $PT(N \backslash C, \emptyset, \emptyset, N \backslash C)$.

Obtaining Facets of PT using Superadditive Lifting

Proposition: Let $C \subseteq N$ and T_0, T_1 be a partition of $N \backslash C$. Assume that C is a nontrivial cover in $PT(N \backslash C, \emptyset, \emptyset, N \backslash C)$, then the inequality

$$\sum_{j \in C} \min\{a_j, \mu\} x_j + \sum_{j \in N \backslash C} \alpha_j x_j + \sum_{j \in C} a_j y_j + \sum_{j \in N \backslash C} \beta_j y_j$$
$$\leq \sum_{j \in C} (\min\{a_j, \mu\} + a_j) - \mu + \sum_{j \in N \backslash C} \beta_j$$

with

$$\alpha_j = \begin{cases} \Phi(a_j) & \text{for} \quad j \in T_1 \\ 0 & \text{for} \quad j \in T_0 \end{cases} \quad and \ \beta_j = \begin{cases} \frac{\mu a_j}{\mu + \sigma(a_j)} & \text{for} \quad j \in T_1 \\ 0 & \text{for} \quad j \in T_0 \end{cases}$$

where

$$\Phi(a) = \begin{cases} -\mu & \text{if} \quad a \leq -\mu \\ (-1+l)\mu - (A_{l-1} - a) & \text{if} \quad A_{l-1} - \mu < a \leq A_{l-1} \\ (-1+l)\mu & \text{if} \quad A_{l-1} < a \leq A_l - \mu \\ +\infty & \text{if} \quad A_n - \mu < a \end{cases}$$

where A_l is the lth largest coefficient of the cover and

$$\sigma(a) = \begin{cases} 0 & \text{if} \quad a \leq A_1 - \mu \\ \max\{\sigma \,|\, \Phi(a - \sigma)\} = \Phi(a)\} & \text{if} \quad a > A_1 - \mu \end{cases}$$

is facet-defining for PT.

Example

Consider

$$19x_1y_1 + 17x_2y_2 + 15x_3y_3 + 10x_4y_4 \leq 20$$

The set $C = \{1, 3\}$ is a cover with excess 14. The cover inequality

$$14x_1 + 14x_3 + 19y_1 + 15y_3 \leq 48$$

is facet-defining for $PS(\bar{C}, \emptyset, T, \bar{T})$ where $\bar{C} = N \backslash C$, $\bar{T} \cup T = \bar{C}$ and $\bar{T} \cap T = \emptyset$. This cover inequality can be lifted into the following inequalities

$$14x_1 + 14x_3 + 19y_1 + 15y_3 + 0 \leq 48 + 0$$
$$14x_1 + 14x_3 + 19y_1 + 15y_3 + 12x_2 + 17y_2 \leq 48 + 17$$
$$14x_1 + 14x_3 + 19y_1 + 15y_3 + 5x_4 + 10y_4 \leq 48 + 10$$
$$14x_1 + 14x_3 + 19y_1 + 15y_3 + 12x_2 + 17y_2 + 5x_4 + 10y_4 \leq 48 + 27$$

which are facet-defining for PS.

Obtaining Facets of PT using Superadditive Lifting

Comments

1. The function Φ given above is the lifting function and is superadditive.

2. It is possible to build an exponential family of inequalities from this lifted cover inequality

   (a) The choices of α_i and β_i given is one choice that give two additional tight points.

3. This result can also be used to derive approximate lifting schemes

Another Family of Strong Inequalities for PT

Definition: Given $K \subseteq N$, we say that K is a clique for PT if

$$a_i + a_j > d, \ \forall i \neq j \in K$$

Proposition: Let $K \subseteq N$. Assume that K is a clique in $PT(N \backslash K, \emptyset, N \backslash K, K)$, then the inequality

$$\sum_{j \in K} x_j \leq 1$$

is a facet of $PT(N \backslash K, \emptyset, N \backslash K, K)$.

Another Family of Strong Inequalities for PT

Using lifting for y_j for $j \in K$ and lifting for $(x_j, y_j) \in N\backslash K$, we obtain:

Proposition:

Let $K \subseteq N$. Assume that K is a clique in $PT(N\backslash K, \emptyset, N\backslash K, K$. Then the inequality

$$\sum_{j \in K} x_j + \sum_{j \in K} \frac{a_j}{a_j - d + \min_K\{a \,|\, a \neq a_j\})} y_j \leq 1 + \sum_{j \in K} \frac{a_j}{a_j - d + \min_K\{a \,|\, a \neq a_j\})}$$

is facet-defining for PT.

Remark:

1. Note that the function $F(a_j) = \frac{1}{a_j - d + \min_K\{a \,|\, a \neq a_j\}}$ is superlinear

2. Some form of superlinearity also holds for nonlinear programs

Example

Consider

$$19x_1y_1 + 17x_2y_2 + 15x_3y_3 + 10x_4y_4 \leq 20$$

The set $K = \{1,2,3,4\}$ is a clique. The clique inequality

$$x_1 + x_2 + x_3 + x_4 \leq 1$$

is facet-defining for $PS(\emptyset,\emptyset,\emptyset,N)$. This clique inequality can be converted by lifting the continuous variables into the following inequality

$$x_1 + x_2 + x_3 + x_4 + \frac{19}{9}y_1 + \frac{17}{7}y_2 + \frac{15}{5}y_3 + \frac{10}{5}y_4 \leq 1 + \frac{19}{9} + \frac{17}{7} + \frac{15}{5} + \frac{10}{5}$$

or equivalently

$$63x_1 + 63x_2 + 63x_3 + 63x_4 + 133y_1 + 153y_2 + 189y_3 + 126y_4 \leq 664$$

which is facet-defining for PS.

Was there any point to this superadditive lifting?

Part II.3:

The Strengh of Lifted Inequalities for Single Constraint Mixed Integer Bilinear Knapsack Set

An Equivalent Integer Programming Formulation for PT

Linearization 1

Let T be the bilinear mixed integer knapsack set defined above. Then

$$T = proj_{(x,y)}\{(x,y,z) \in \{0,1\}^n \times [0,1]^n \times [0,1]^n \mid \sum_{j=1}^{n} a_j z_j \le d, \quad x_j + y_j - 1 \le z_j, \forall j = 1,\dots,n\}$$

Linearization 2

Let T be the bilinear mixed integer knapsack set defined above. Then

$$T = \{(x,y) \in \{0,1\}^n \times [0,1]^n \mid \sum_{j \in T} a_j x_j + \sum_{j \in T} a_j y_j \le d + \sum_{j \in T} a_j, \forall T \subseteq N\}$$

An Equivalent Integer Programming Formulation for PT

Comments

1. Linearization 2 is a 0-1 mixed integer program with many constraints. Inequalities could be obtained through aggregation of the rows.

2. Assume that we consider creating all valid inequalities from each row and/or aggregation of rows of this formulation. Can the lifted cover cuts be obtained this way, i.e. is it in the "aggregation"-closure?

3. This closure is not directly related to Chvátal-Gomory and split closures

Strong Rank-1 Inequalities

Definition:

Let U be the feasible region of a mixed integer program:

$$U = \{x \in \mathbb{Z}^n \times \mathbb{R}^n \,|\, Ax \leq b\}$$

and an inequality

$$\sum_{j=1}^{m+n} \alpha_j x_j \leq \delta(*)$$

valid for U. We say that (*) is a strong rank-1 inequality if there exist $u \in \mathbb{R}^n_+$ such that

$$\sum_{j=1}^{n+m} u_j = 1$$

such that (*) is a valid inequality for

$$P(u) = conv\{x \in \mathbb{Z}^m \times \mathbb{R}^n \,|\, uAx \leq ub\}$$

High Rank Certificate

Definition: We say that a set of points $\{x^i\}_{i=1,\dots,k} \in \mathbb{Z}^m \times \mathbb{R}^n$ form a high-rank certificate for (*) if

1. $\sum_{j=1}^{m+n} \alpha_j x_j^k > \delta$

2. $\forall u \in \mathbb{R}_+^{m+n}$ such that $\sum_{j=1}^{m+n} u_j = 1$ then there exists $k(u) \in \{1, \dots, k\}$ such that

$$uAx^k(u) \leq ub$$

Proposition: $\{x^i\}_{i=1,\dots,k} \in \mathbb{Z}^m \times \mathbb{R}^n$ is a high-rank certificate for (*) if and only if

1. $\sum_{j=1}^{m+n} \alpha_j x_j^k > \delta$

2. $conv\{x^1, \dots, x^k\} \cap LPrelax(U) \neq \emptyset$

Lifted Cover Cuts for BMIKP are not Strong Rank-1

Example : Consider

$$PT = \{(x,y) \in \{0,1\}^3 \times [0,1]^3 \mid \sum_{j=1}^{3} a_j x_j y_j \leq a_2 + a_3 - \mu\}$$

where (i) $a_1 > a_2 \geq a_3 > \mu > 0$, (ii) $a_1 < a_2 + a_3 - \mu$. Then the inequality

$$x_1 + x_2 + x_3 + \frac{a_1}{\mu + a_1 - a_2} y_1 + \frac{a_2}{\mu} y_2 + \frac{a_3}{\mu} y_3 \leq 1 + \frac{a_1}{\mu + a_1 - a_2} + \frac{a_2}{\mu} + \frac{a_3}{\mu}$$

is a facet of PT that does have high rank-1 since the following two points

1. $(x_1 = 1, y_1 = \frac{a_1 - \mu - \sigma}{a_1} + \epsilon, x_2 = 0, y_2 = 1, x_3 = 1, y_3 = 1)$
2. $(x_1 = 0, y_1 = \frac{a_1 - \mu - \sigma}{a_1} + \epsilon, x_2 = 1, y_2 = 1, x_3 = 1, y_3 = 1)$

form a high-rank certificate for $\epsilon > 0$ sufficiently small.

Towards the Next Step...

Remarks:

1. We have derived methods to generate linear cuts for mixed integer nonlinear programs

2. Some of these cuts cannot be obtained easily using linearizations.

3. For most problems, it might be necessary to add nonlinear constraints

4. We want to describe methods to use the results on linear cuts to generate nonlinear cuts

Part III:

Example 2: Nonlinear Disjunctive Sets

Goal of Part III

Illustrate that:

It is possible to derive nonlinear cutting planes

Illustration on:

Simple Disjunctive Sets (why?)

A General Procedure

Steps

1. Reduce the dimension of the set studied by fixing some of the variables so that the convex hull of the restriction can be described by a parameterized (in θ) family of linear inequalities

2. Lift this family of linear inequalities to reintroduce the fixed variables

3. For the obtained family of linear inequalities, determine for each point of the set studied, the value of θ that create the tighest constraint. Doing so, θ is expressed as a function of the variables of the set studied, $\theta(x)$

4. Substitute that function $\theta(x)$ in the parameterized family of inequalities to obtain a nonlinear cut.

Deriving Nonlinear Cuts for Mixed Integer Programs: Applications

Advantages

1. Can be applied to many problems
2. Only require the derivation of linear inequalities
3. Systematic way to obtain convex hulls in the initial space of variables

Limitations

1. Might be difficult to solve the lifting problems analytically

Illustrations

1. We derive the convex hull of a 3D nonpolyhedral BKP
2. We derive the convex hull of two disjunctive sets

Obtaining the Convex Hull of a Simple Bilinear Knapsack Set

We now consider the set

$$V = \{(x, y, z) \in [0, 1]^3 \,|\, axy + bz \geq 1\}$$

where (i) $a \geq 1$ and (ii) $b < 1$.

Proposition:
The convex hull PV of V is given by

$$PV = \{(x, y, z) \in [0, 1]^3 \,|\, \sqrt{axy} + (1 - \sqrt{1 - b})z \geq 1 (*)\}$$

Remark:
For the case $a = 2$, $b = \frac{1}{2}$, the point $(1, \frac{1}{4}, \frac{1}{4})$ satisfies both the relaxations

$$2y + \frac{1}{2}z \geq 1$$
$$2x + \frac{1}{2}z \geq 1.$$

However, it does not satisfy (*).

Obtaining the Convex hull of a Simple Bilinear Knapsack Set

Methodology:

1. Fix $z = 0$ and $x = \theta$ where $\theta \geq \frac{1}{a}$. We obtain $a\theta y \geq 1(*)$

2. Lift the inequality (*) with respect to x to obtain the inequalities $\frac{1}{\theta}x + a\theta y \geq 2$ for all $\theta \in [\frac{1}{a}, 1]$

3. Lift all of these inequalities with respect to z to obtain

$$\frac{1}{\theta}x + a\theta y + \gamma z \geq 2 \quad \text{where} \quad \gamma = \max \frac{2 - \frac{1}{\theta}x - a\theta y}{z} \quad \text{s.t. } axy + bz \geq 1, z > 0$$

4. We obtain, for all $\theta \in [\frac{1}{a}, 1]$,

$$\frac{1}{\theta}x + a\theta y + 2[1 - \sqrt{1-b}]z \geq 2$$

Obtaining the Convex hull of a Simple Bilinear Knapsack Set

Methodology:

1. We obtain, for all $\theta \in [\frac{1}{a}, 1]$,

$$\frac{1}{\theta}x + a\theta y + 2[1 - \sqrt{1-b}]z \geq 2(**)$$

2. For any given (x, y, z), the strongest of these inequality is the one for which the value of

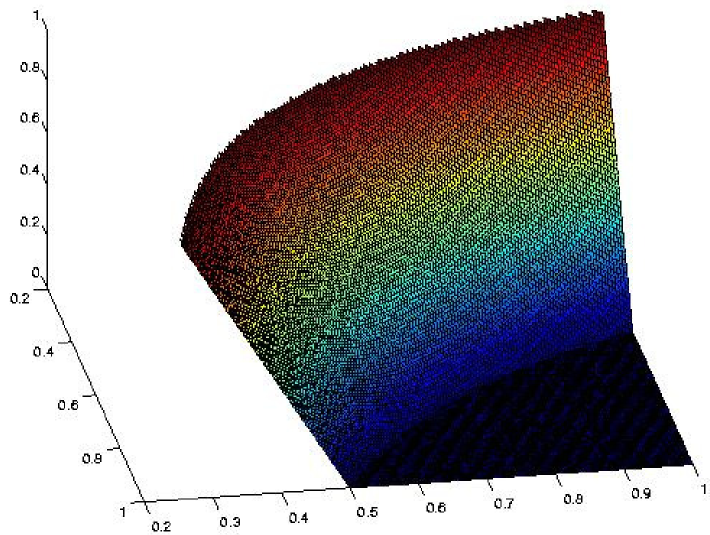
$$\frac{1}{\theta}x + a\theta y$$

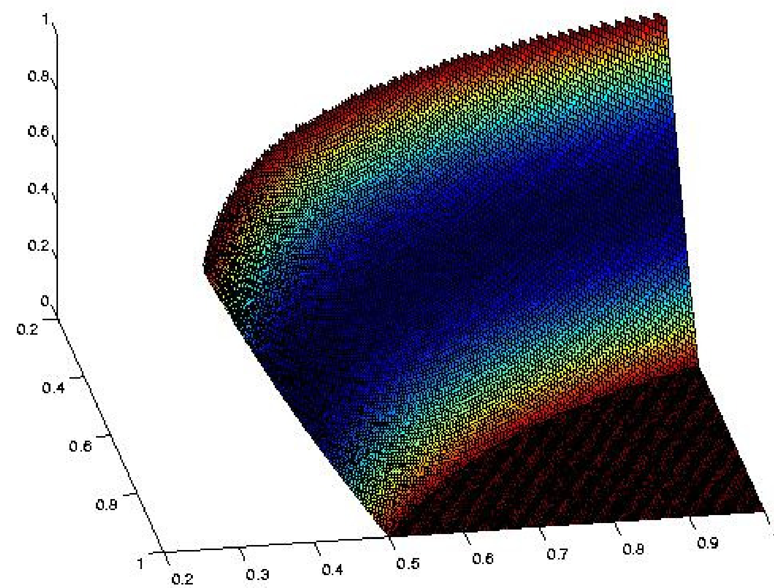
is minimum.

3. Using KKT conditions, we obtain that $\theta(x, y, z) = \sqrt{\frac{x}{ay}}$

4. Subsituting in (**), we obtain the result.

Obtaining the Convex hull of a Simple Bilinear Knapsack Set





Obtaining Convex Hulls of Disjunctive Sets: An Example

Example: Consider the disjunctive set

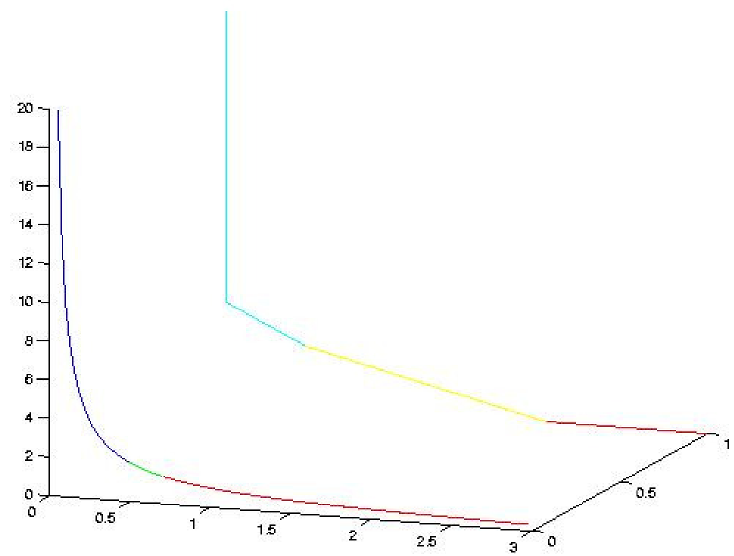
$$S = \{(x, y, z) \in \mathbb{R}^3_+ \quad | \quad \begin{array}{l} x > 0 \\ z \geq 1/x \text{ when } y = 0 \\ z \geq \max\{5 - 4x, 4 - 2x\} \text{ when } y = 1\} \end{array}$$

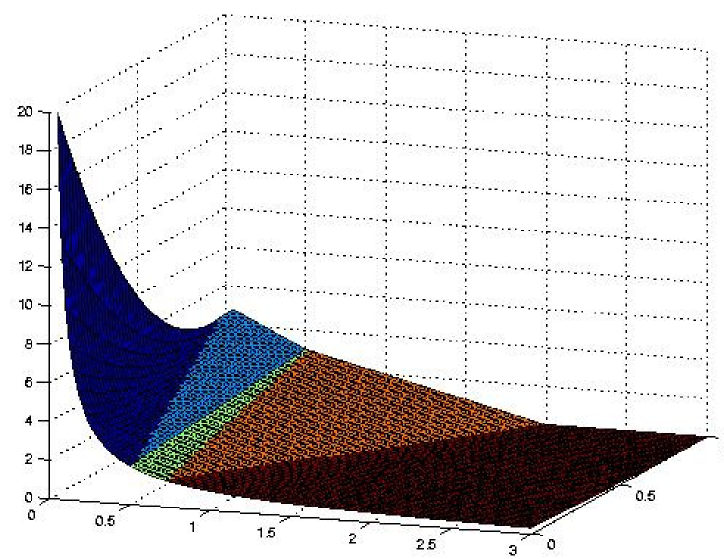
The convex hull of S is given by

$$conv(S) = \{(x, y, z) \in \mathbb{R}^3_+ \quad | \quad \begin{array}{l} z \geq \dfrac{1 + y^2 - 2y + 5xy}{x} \text{ if } 2x + y \leq 1 \\ z \geq 4 - 4x + y \text{ if } 2x + y \geq 1 \& x \leq 0.5 \\ z \geq \dfrac{2 - 4y + 6xy - y^2}{2x - y} \text{ if } x \geq 0.5 \& y + \alpha x \leq \alpha \\ z \geq 2\sqrt{2} - 2x - \beta y \text{ if } y + \alpha x > \alpha \& y - \dfrac{\beta}{2} x \geq 1 - \beta \\ z \geq \dfrac{(1 - y)^2}{x - 2y} \text{ if } y - \dfrac{\beta}{2} x < 1 - \beta\} \end{array}$$

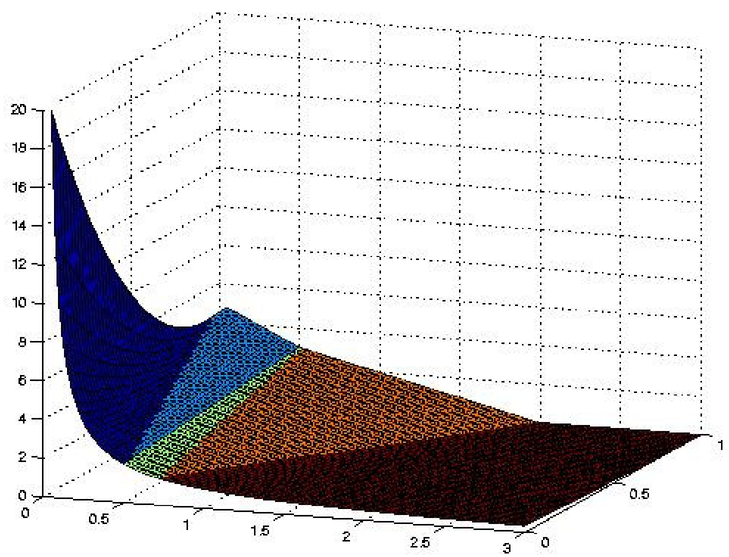
where $\alpha = \frac{\sqrt{2}}{\sqrt{2}-1}$, $\beta = (2\sqrt{2} - 4)$

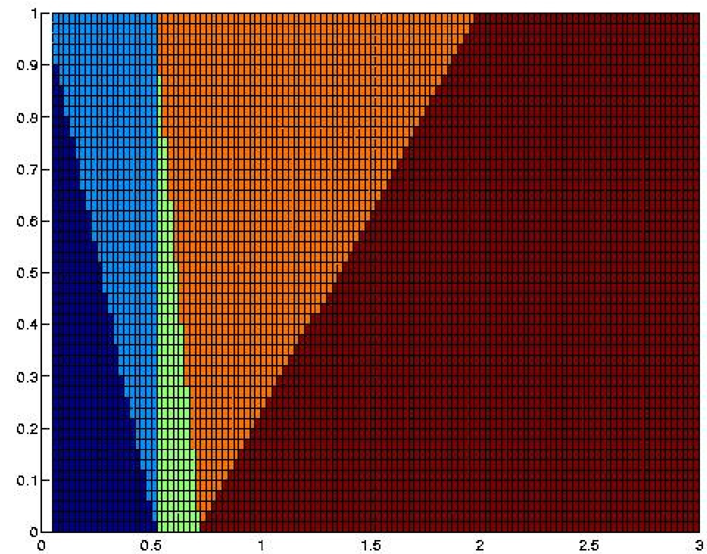
Obtaining Convex Hulls of Disjunctive Sets: An Example





Obtaining Convex Hulls of Disjunctive Sets: An Example





Obtaining the Convex Hull of Disjunctive Sets: A Result

Proposition: Let $F(x), G(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ be convex differentiable decreasing function. Also assume that their derivatives $f(x)$ and $g(x)$ are invertible. Define the set

$$S = \{(x,y,z) \in R^3_+ \quad | \quad \begin{array}{l} z \geq F(x) \text{ when } y = 0 \\ z \geq G(x) \text{ when } y = 1\} \end{array}$$

Then the convex hull of S

$$conv(S) = \{(x,y,z) \in R^3_+ \quad | \quad \begin{array}{l} z \geq \theta(x - f^{-1}(\theta)(1-y) - g^{-1}(\theta)y) \\ \quad + F(f^{-1}(\theta))(1-y) + G(g^{-1}(\theta)y)\} \end{array}$$

where θ is a solution to $x = f^{-1}(\theta)(1-y) + g^{-1}(\theta)y$

Obtaining the Convex Hull of Disjunctive Sets: An Example

Example: Consider the disjunctive set

$$S = \{(x, y, z) \in \mathbb{R}^3_+ \quad | \quad z \geq e^{-x} \text{ when } y = 0$$
$$z \geq e^{-2x} \text{ when } y = 1\}$$

The convex hull of S is given by

$$conv(S) = \{(x, y, z) \in \mathbb{R}^3_+ \quad | \quad z \geq (1 - \frac{y}{2})e^{\frac{\frac{ln2}{2}y - x}{1 - \frac{y}{2}}}\}$$

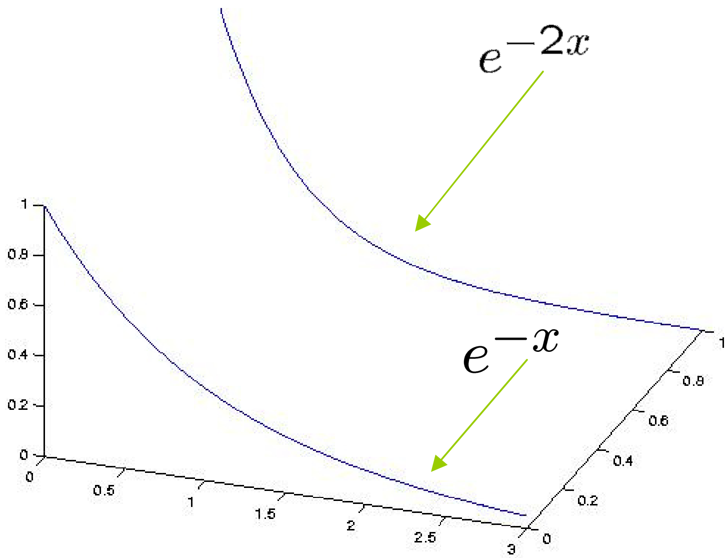
Proof: From previous Proposition, we have

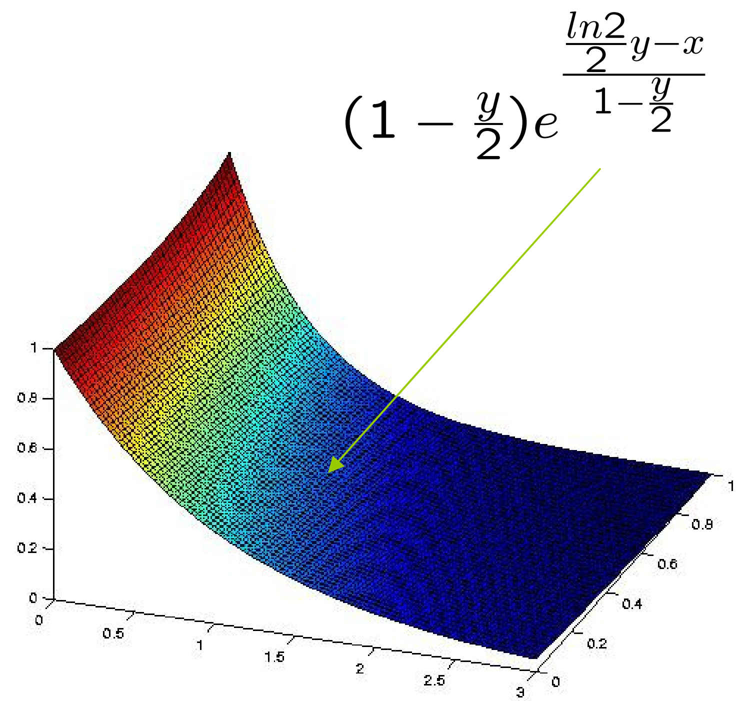
◇ $F(x) = e^{-x}$, $f(x) = -e^{-x}$, $f^{-1}(x) = -ln(-x)$

◇ $G(x) = e^{-2x}$, $g(x) = -2e^{-2x}$, $g^{-1}(x) = -\frac{1}{2}ln(\frac{x}{2})$

◇ $\theta = -e^{\frac{\frac{ln2}{2}y - x}{1 - \frac{y}{2}}}$

Obtaining the Convex Hull of Disjunctive Sets: An Example





Some Comments

Comments:

1. The same ideas carry over to disjunctive sets over more than two disjunctions
2. Similar results can be obtained starting from nonconvex or from nondifferentiable functions
3. When the dimension of the domain of the disjunctive sets is larger than one, we can extend theoretically extend the results ... But the analytical derivation become hard

Futures:

1. Investigate how to derive numerical variants of the approach
2. Investigate how to derive symbolic variants of the approach

Part IV:

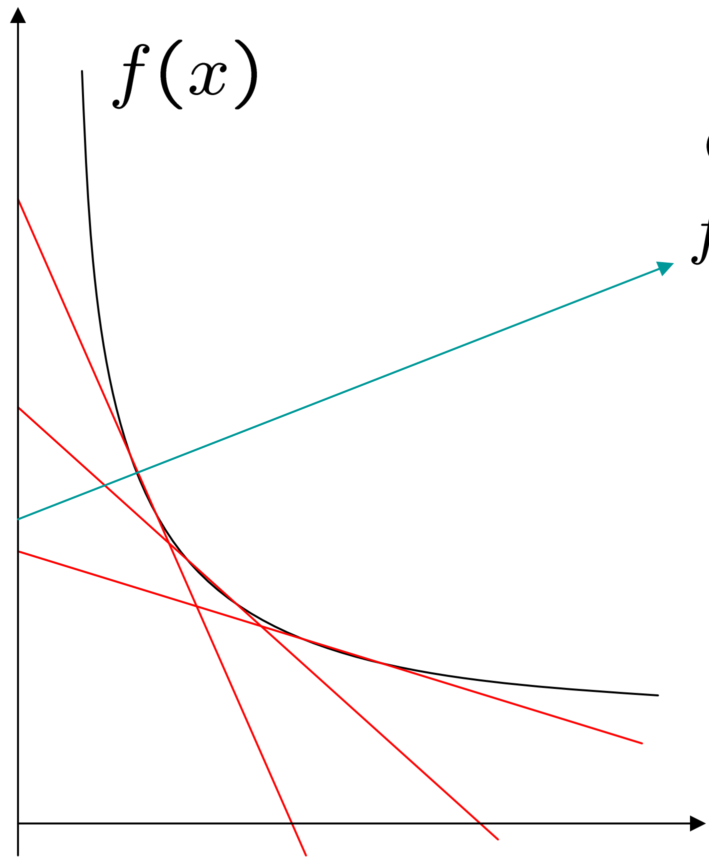
Some Theoretical Results

Goal of Part IV

Illustrate that:

There is a set of common concepts that relate all of these results

Generalizing the Theory…



Conjugate:

$f^*(s) := sup_x\{< s, x > -f(x)\}$

Generalizing the Superadditive Lifting Theory...

- In Integer Programming, superadditive lifting guarantees that sequence independence is obtained
- Lifting functions are closely related to conjugate functions
- Therefore, sequence independent lifting should have an interpretation in terms of conjugates
- In order to obtain independence, the values of the conjugate should be driven mainly by its value on the axes
- This naturally leads to superadditivity of a function related to the conjugate

Application: Sequence Independent lifting for single-constraint problems

- Consider $S = \{(x_0, \ldots, x_m) \,|\, f(x_0, \ldots, x_m) \le a\}$
- Define $\gamma(x_1, \ldots, x_m) = -\sup_{x_0}\{< \alpha_0, x_0 > \,|\, f(x_0, \ldots, x_m) \le a\}$
- Then $\alpha_0 x_0 \le \beta$ is valid for S where $x_i = 0$ for $i = 1, \ldots, m$ if $\beta = \gamma(0, \ldots, 0)$
- Define the perturbation ("lifting") function $P(w) = \beta - \sup_{x_0}\{\alpha_0 x_0 \,|\, f(x_0, 0, \ldots, 0) \le a - w\}$
- Finally define $g(x_0, \ldots, x_m) = f(x_0, 0, \ldots, 0) - f(x_0, \ldots, x_m)$ and $\theta(x_1, \ldots, x_m) = P(g(x_0, \ldots, x_m))$.

Application: Sequence Independent lifting for single-constraint problems

Theorem:
If there exists h_1 and h_2 such that

1. $g(x_0,\ldots,x_m) \geq h_1(\Pi_{i=1}^m g(x_0,0,\ldots,0,x_i,0,\ldots,0))$

2. $P(h_1(\Pi_{i=1}^m g(x_0,0,\ldots,0,x_i,0,\ldots,0))$
$\geq h_2(\Pi_{i=1}^m P(g(x_0,0,\ldots,0,x_i,0,\ldots,0))))$

where h_2 is monotone convex. Then the convex set

$$\bar{S} = \{(x_0,\ldots,x_m) \quad | \quad \begin{array}{l} \beta - \alpha_0 x_0 \geq h_2(z_1,\ldots,z_m) \\ z_i \geq \bar{\theta}_i(0,\ldots,0,x_i,0,\ldots,0) \text{ for } i = 1,\ldots,m\} \end{array}$$

outer approximates S if $\bar{\theta}_i(0,\ldots,0,x_i,0,\ldots,0)$ is a convex underestimator for $\theta(0,\ldots,0,x_i,0,\ldots,0)$

Part V:

Conclusion

Conclusion & Future Work

1. We showed that MIP lifting techniques can be fruitful in non-linear programming

2. We introduced techniques that can be used to make them computationally efficient

3. For mixed integer polyhedral sets, these techniques can be used to obtain quickly inequalities that cannot be obtained directly from linear IP reformulations

4. Although originally designed for linear cuts, the theory can be used to generate nonlinear cuts and convex hulls in the space of the original variables

5. We presented a generalization of superadditive lifting to non-linear programs

6. This is only the tip of the iceberg, much more need to be done to make the approach practical