Computing Forecast Horizons: An Integer Programming Approach

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Agenda

- Introduction to forecast horizons.
- Concept of a discrete forecast horizon.
- Using integer programming to compute forecast horizons.
- Analysis of discrete forecast horizons.
- Future potential.

Forecast Horizons

- Consider a multi-period decision-making problem.
- The number of periods, t, for which a decision has to made in the current period is called a *decision horizon*.
- An integer, T, is referred to as a *forecast horizon* corresponding to the decision horizon t if the data beyond period T do not influence the optimum decision for the first t periods for *any* problem with terminal time longer than T.

Forecast Horizons

- In other words: For any N-period problem, $N \geq T + 1$, there exists at least one optimum solution with the same decision in the first period.
- If T is a forecast horizon, then so is every $N \geq T$.
- Interesting question: What is the minimal forecast horizon?
- Notation: forecast horizon: FH^0, minimal forecast horizon: FH$^0_{min}$.

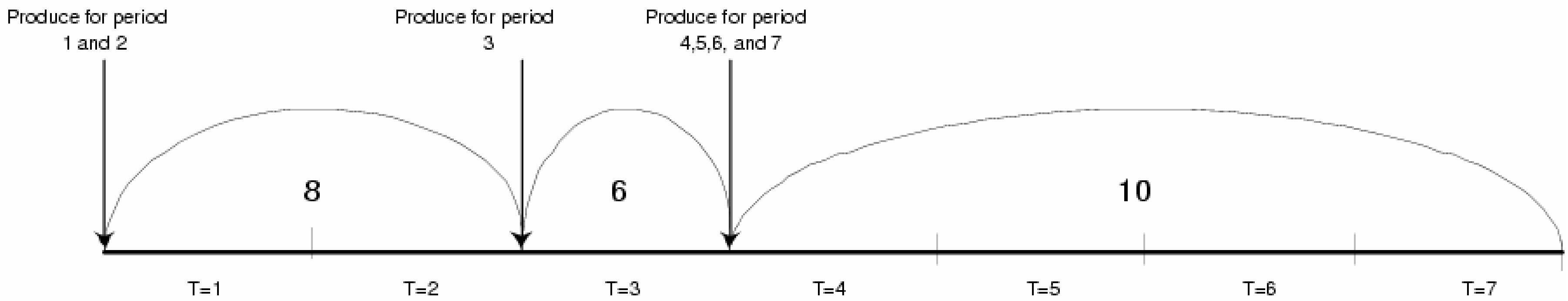
The Dynamic Lot-Size Problem (DLSP)

- We use the DLSP to illustrate concepts.
- Dynamic Version of the Economic Lot Size Model.
  - T-period production problem with known demand, holding and setup costs.
  - Objective is to find a production plan such that all demands are met at a minimum cost.

DLSP - an example:

Period:	1	2	3	4	5	6	7
Demand:	5	3	6	2	4	3	1

set up cost = 2; holding cost = 1;



Cost of the three setups: $3 \times 2 = 6$

Holding cost: $(3 \times 1) + (4 \times 1) + (3 \times 2) + (1 \times 3) = 16$; total cost = 22.

Definitions and decision variables

T = the problem horizon, i.e., the number of periods.

k = setup cost for production.

h = holding cost per unit per period.

d_j = demand in period j, $j = 1, 2, ..., T$.

Q_j = quantity produced in period j.

$$X_j = \begin{cases} 1 & \text{if a setup is required in period } j; \\ 0 & \text{otherwise;} \end{cases}$$

I_j = inventory at the end of period j.

A Formulation for DLSP

Minimize $$\sum_{j=1}^{T}[kX_j + hI_j]$$

subject to:

$$Q_j \le (\sum_{r=j}^{t} d_r)X_j \qquad 1 \le j \le T$$

$$I_j = I_{j-1} + Q_j - d_j \qquad 1 \le j \le T$$

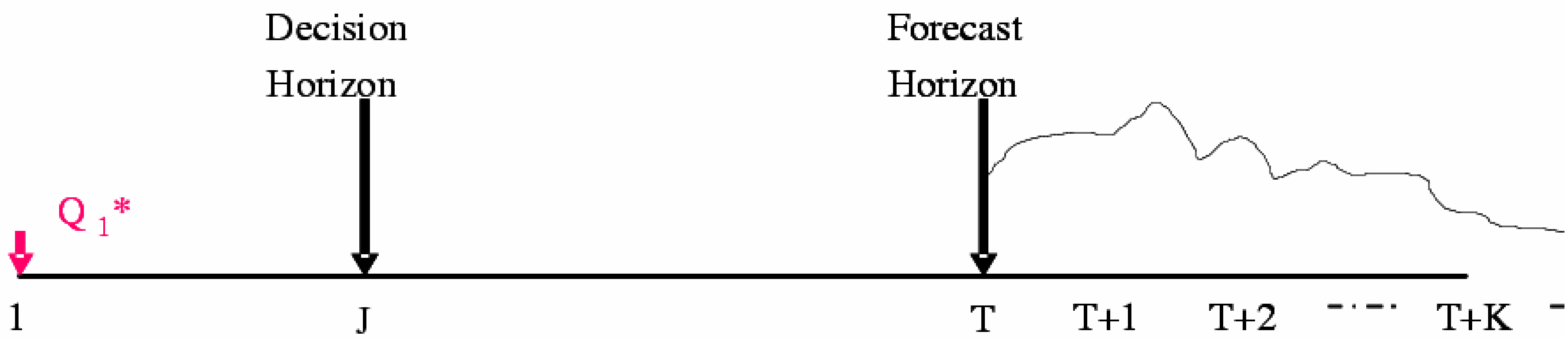
$$I_0 = 0$$

$$I_T = 0$$

$$I_j, Q_j \ge 0$$

$$X_j \in \{0, 1\}$$

Forecast and decision horizon



- T is a forecast horizon *iff at least one optimum solution* to every $(T+F)$-period problem $(F \geq 0)$ has the same first period production Q_1^*.

Advantages of forecast horizons

- Forecasting farther into the future is expensive and is error-prone.
- Reducing the computational burden of solving a larger horizon problems.

Applications

Forecast horizons have been investigated for many real-world dynamic decision problems:

- Production planning -*Aronson et al. (1984)*
- Cash management -*Chand and Morton (1982)*
- Inventory management -*Federgruen and Tzur (1996)*
- Machine replacement -*Chand and Sethi (1982)*
- Plant location -*Daskin et al. (1992)*
- Sequencing and scheduling -*Rempala (1989)*

Algorithms to detect forecast horizons

- Dynamic Programming and Optimal Control are the most frequently used methods to detect forecast horizons.
- IP has largely been ignored as an approach for computing forecast horizons.

Algorithms to detect forecast horizons

- Forward Algorithm.
  (H.M. Wagner and T.M. Whitin, MS 1958)
  - An efficient procedure for solving longer horizon problems.
  - This was the first attempt at computing forecast horizons.
- Necessary conditions.
  (R.A. Lundin E. Morton, OR 1975)
- Necessary and Sufficient conditions.
  (S. Chand and E. Morton, MS 1982)

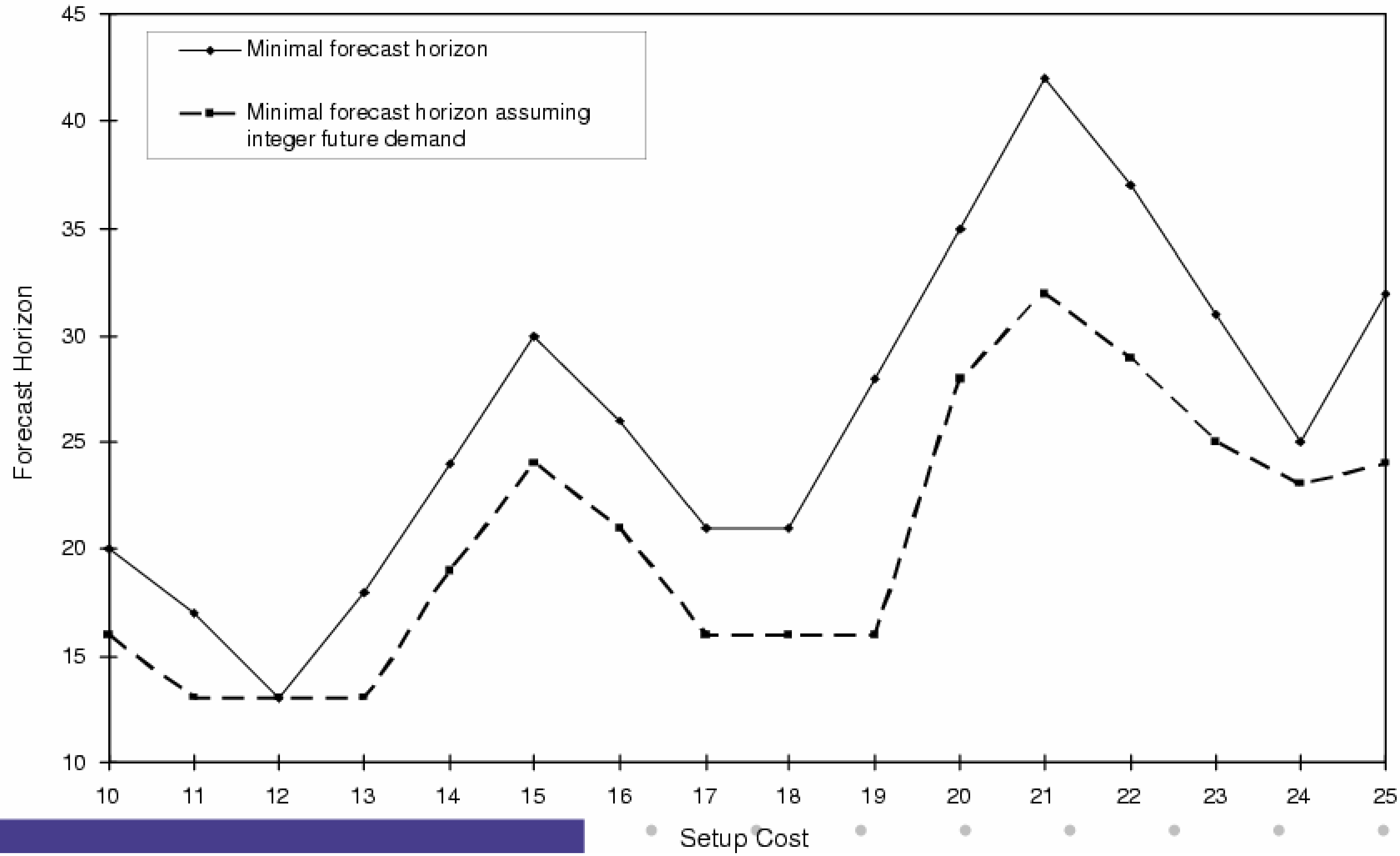
Introduction to discrete forecast horizons

- The classical notion of a forecast horizon places no restrictions on the future data.
- The DLS model of Wagner and Whitin accommodates the possibility that a future demand could take any non-negative value.
- In practice, the context of the problem being investigated often allows us to specify additional characteristics of future demands.

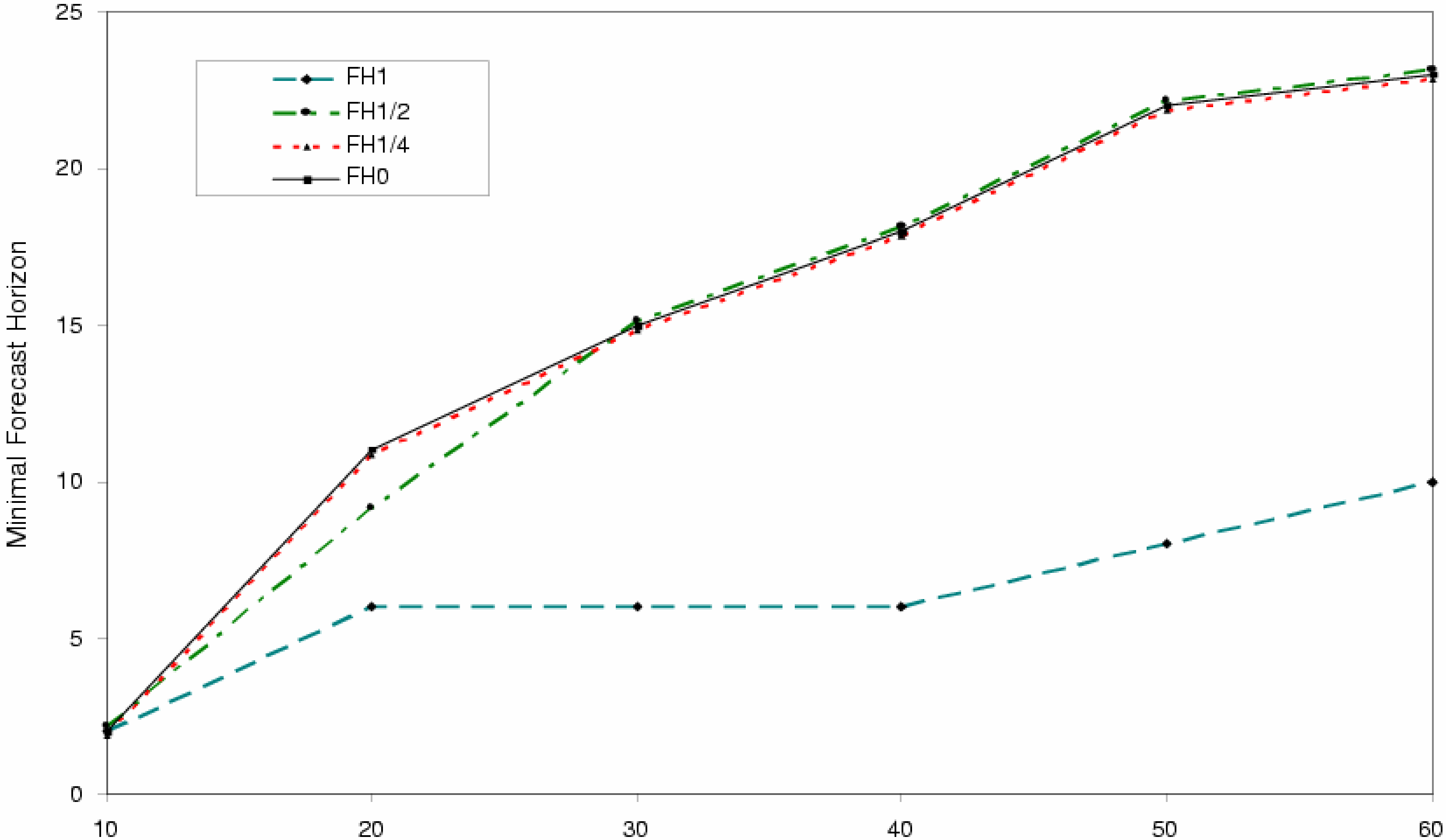
Introduction to discrete forecast horizons

- Typically, demand realizations obey a well-defined granularity.
  - A car manufacturer that needs to consider only integer valued demands.
  - An oil refinery quantifies demand in the thousands of gallons.
- In these cases, the business is not interested in considering demands that fall in between these discrete values.
- FH^q: a forecast horizon assuming future demands are multiples of $q \in \Re_+$.

Reduction in the length of the minimal horizon

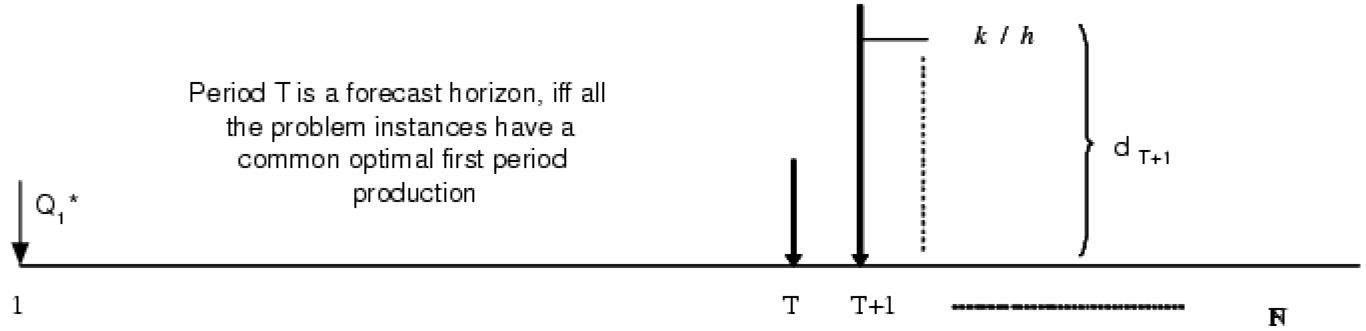


Convergence of FH^q to FH^0: an example



A characterization for a period T to be a FH^0

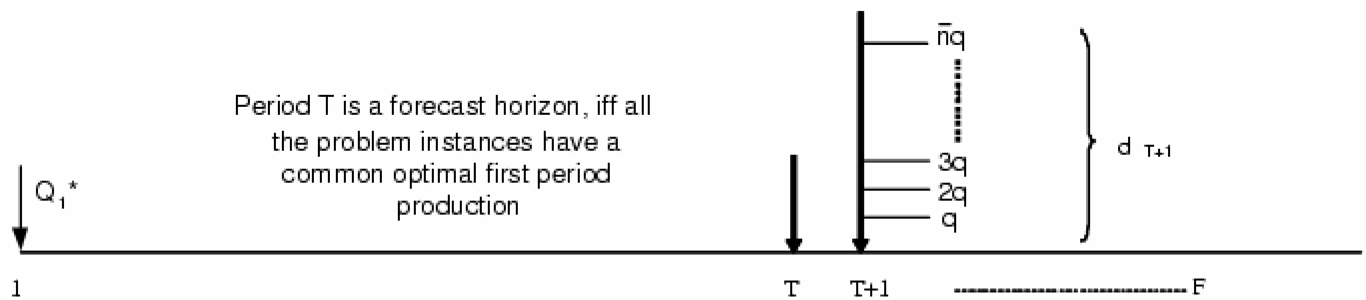
- Every optimum production plan in the first T periods of an N-period problem, $N \geq T+1$, can be found by solving the $(T+1)$-period problem with demands $d_i, i = 1, ..., T$, in the first T periods and some appropriately selected demand α in period $T+1$. *(Chand and Morton, 1982)*.



$d_{T+1} \in \{[0, \frac{k}{h}] : \Re^1\}$

Characterization for FH^q

T is a forecast horizon assuming that future demands are integer multiples of q (i.e., T is an FH^q) if, and only if, $\cap_{\alpha \in \{0,q,2q,\ldots,\overline{n}q\}} \mathcal{Q}_1(\alpha) \neq \emptyset$, where $\overline{n}$ is the smallest integer such that $\overline{n}q \geq \frac{k}{h}$.



$$d_{T+1} \in \{0, q, 2q, \ldots \bar{n}q\}$$

Definitions and decision variables

$M_{T+1}(\alpha)$: Optimal cost for a $(T+1)$-period problem when the demand in period $T+1$ is α. Note that $\alpha = 0$ is essentially the T-period problem.

Λ_q : the set of demands for period $T+1$, indexed by α, $\Lambda_q \in \{n \times q; n = 0, 1, 2, \ldots.\overline{n}\}$.

$$X_j^\alpha : \begin{cases} 1 & \text{if a setup is required in period } j \text{ in} \\ & \text{the problem with period } T+1 \text{ demand } \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

Q_j^α : the quantity produced in period j in the problem with period $T+1$ demand α.

I_j^α : the inventory carried over from period j to $j+1$ in the problem with period $T+1$ demand α.

Checking if T is an FH^q

Find a feasible solution to the following set of constraints:

$$\sum_{j=1}^{T+1} kX_j^\alpha + \sum_{j=1}^{T+1} hI_j^\alpha = M_{T+1}(\alpha) \qquad \alpha \in \Lambda_q$$

$$\left(\sum_{r=j}^{T+1} d_r\right)X_j^\alpha \geq Q_j^\alpha \qquad 1 \leq j \leq T+1,\ \alpha \in \Lambda_q$$

$$I_{j-1}^\alpha + Q_j^\alpha - d_j - I_j^\alpha = 0 \qquad 1 \leq j \leq T,\ \alpha \in \Lambda_q$$

$$I_{j-1}^\alpha + Q_j^\alpha - \alpha - I_j^\alpha = 0 \qquad j = T+1,\ \alpha \in \Lambda_q$$

$$Q_1^\alpha - Q_1^{\alpha'} = 0 \qquad \alpha \neq \alpha';\alpha, \alpha' \in \Lambda_q$$

$$I_0^\alpha, I_{T+1}^\alpha = 0 \qquad \alpha \in \Lambda_q$$

$$I_j^\alpha, Q_j^\alpha \geq 0 \qquad 1 \leq j \leq T+1,\ \alpha \in \Lambda_q$$

$$X_j^\alpha \in \{0,1\} \qquad 1 \leq j \leq T+1,\ \alpha \in \Lambda_q$$

Advantages of the discreteness assumption

- Discretization of future demands allows us to better exploit a characterization for forecast horizons.
- Checking whether a period T is a forecast horizon can be posed as a feasibility question in a system of linear inequalities with 0-1 variables.
- This approach does not depend explicitly on the structural properties of the problem being studied. Consequently, it becomes relatively easy to extend the approach to investigate a wider class of forecast horizons.

The relationship between FH^q and FH^0

- Trivially, $\mathrm{FH}^q_{min} \leq \mathrm{FH}^0_{min}$.
- Interesting question: Conditions under which $\mathrm{FH}^q_{min} = \mathrm{FH}^0_{min}$.
- There are two considerations that enable us to derive such conditions:
  - The behavior of the optimal cost with respect to the granularity q of a discrete forecast horizon.
  - The sensitivity of the last production period to perturbations in future data.

Relationship between FH^q and FH^0: sample results

- Let T be an FH^q. Let P(α) denote the $T+1$-period problem with demand α in Period $T+1$. For all $\alpha \in \Lambda_q$, if restricting the first-period production for Problem P(α) to a quantity other than an optimal quantity makes its cost at least as large as that of P($\alpha+q$), then T is an FH^0.

Relationship between FH^q and FH^0: sample results

- Let T be an FH^q. If Problems P($\alpha + \delta$), $0 < \delta < q$ have an optimal solution with the same last regeneration period as that of Problem P(α), then we can ignore these problems when checking for a continuous horizon. If this holds for all $\alpha \in \Lambda_q$, then T is an FH^0.

Concept of $(1+\epsilon)$-FH^0

- **Definition:** Period T is an $(1+\epsilon)$-FH^0 if for every N-period problem, $N \geq T+1$ and all vectors of demands $\bar{d}_{T+1}^{N} \in \Re_{+}^{N-T}$, there exists at least one solution with $(Q_1)^*$ as the first period production quantity, and with cost at most $(1+\epsilon)$ times the optimal cost.
- If T is an FH^q, then T is an $(1+\epsilon)$-FH^0, where, in general, ϵ depends on q.

Continuous forecast horizons (FH^0)

- Sufficient condition: T is a forecast horizon if an optimum solution with first-period production, say Q_1^T, to the T-period problem with last production in period L satisfies the following condition: every S-period problem with $L-1 \leq S \leq T$ has an optimum solution with first-period production Q_1^T.
- L: last production period; $L-1$: last regeneration period.
- *Regeneration set*: periods from $L-1$ to T.

Continuous forecast horizons (FH^0)

- The actual regeneration set could be much smaller: *minimal regeneration set*.
- Necessary and Sufficient conditions (Chand and Morton, 86):
  - Identify a minimal regeneration set.
  - Test whether all the planning horizons in it have a common optimum production quantity.
- Efficiently solvable IP formulations.

Extensions of discrete horizons

- Forecast horizons for a capacity-constrained dynamic lot-sizing problem.
- A two-product problem with individual and joint setup costs.
- Development of necessary and sufficient conditions and corresponding IP formulations.
- Satisfactory performance of IP solvers on a wide variety of problems.

Interesting research issues

- Existence of forecast horizons.
- Characterization of demand vectors for which a forecast horizon exists.
- Proving bounds on the magnitude of reduction offered by discrete horizons.
- Exploiting the constraint-based nature of integer programming to compute forecast horizons for a wider range of dynamic decision-making problems.