Robust Portfolio Construction

Presentation to
Workshop on Mixed Integer Programming

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Glossary

Assets (n)

Investable securities (*U*), typically stocks (equities)

Portfolio

Holdings: Initial (*h*), final (*x*) (holdings are represented in % or dollars)

Long holdings: ($i : x_i > 0$)

Short holdings: ($i: x_i < 0$)

Trades

(x-h)

Benchmark

A market portfolio: S&P 500, Russell 1000 (typically market-cap weighted) (*b*)

Budget

The total amount invested (*B*)

Expected Returns (Expected Active Returns)

A vector (α) of expectations of return (in percent), expected return of a portfolio $\alpha' x$ ($\alpha'(x-b)$)

Covariance of Returns

A matrix (*Q*) representing the forecasted covariances of returns

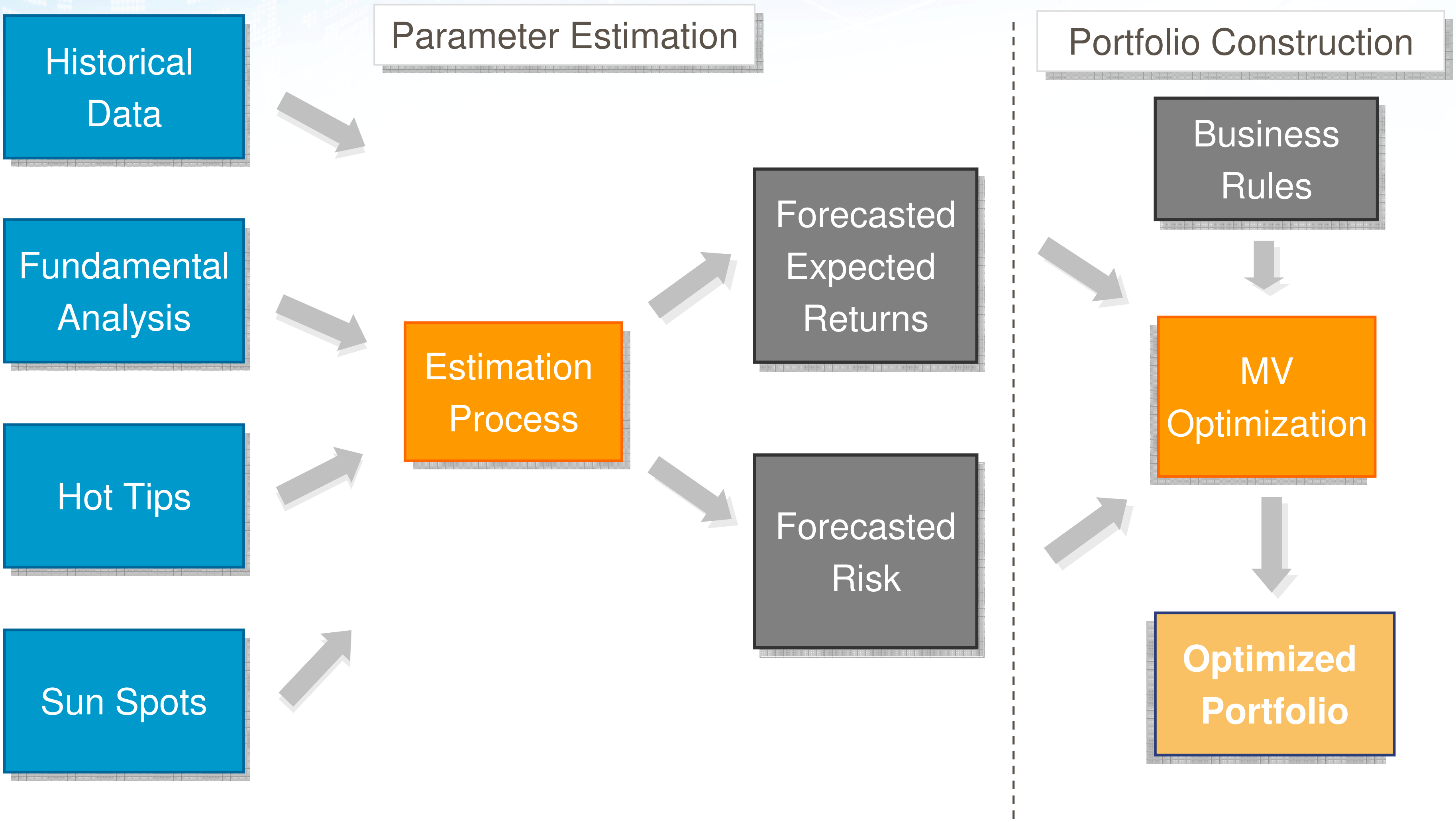
Predicted Risk of a portfolio

$x'Qx$

Predicted Tracking Error

$(x-b)'Q(x-b)$

Roadmap of the Quant Process



Mathematical Models (MVO)

Active Management

$$\text{Max } \alpha^t x - \lambda(x-b)^t Q(x-b)$$

$$\text{st.} \quad \sum_{i \in U} x_i = B$$

$$x_i \geq 0, i \in U$$

Expected Returns

Risk Aversion

Budget

Tracking Error

No Shorting

Why Don't Practitioners Use MVO Extensively?

- Naïve portfolio rules, such as equal weighting, can outperform traditional MVO (Jobson and Korkie)
- Optimal portfolios from MVO are not necessarily well diversified (Jorion) or intuitive (Several authors)
- MV Optimizers have the "Error Maximization Property" . MVO will tend to overweight assets with positive estimation error and underweight assets with negative estimation error (Several authors)
- Unbiased risk and expected return estimators still lead to a biased estimate of the efficiency frontier (Several authors)

Portfolio Managers spend most of their time "cleaning up" the optimal portfolio provided by MVO

Criticisms of MVO

Literature Review: There is an extensive literature that studies the effects of estimation error in classical mean variance optimization

- Jobson and Korkie, “Putting Markowitz Theory to Work”, JPM, 1981 (and related work)
- Jorion. “International Portfolio Diversification with Estimation Risk, Journal of Business, 1985 (and related work)
- Chopra and Ziemba, The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice”, JPM, 1993
- Broadie, “Computing Efficient Frontiers Using Estimated Parameters”, Annals of OR, 1993
- Michaud, “The Markowitz Optimization Enigma: Are Optimized Portfolios Optimal?”, FAJ 1989 and “Efficient Asset Management”, Oxford Univ Press, 1998

How do Practitioners "fix" MVO? (Extensions)

- **Simple (Linear):**
  - Initial holdings (h)
  - Transaction variables ($t = |x - h|$)
  - Limits on holdings/trades ($x \leq u, t \leq v$)
  - Industry/Sector Holdings ($\sum_{i \in S} x_i \leq c$)
  - Active Holdings, Industry/Sector Active Holdings ($|x - b| \leq u$, $\sum_{i \in S} |x_i - b_i| \leq c$)
  - Limits on Turnover, Trading, Buys/Sells ($\sum_{i \in S} t_i \leq c$)

- **Complex (Linear-Quadratic):**
  - Long/Short Portfolios (eliminate $x \geq 0$)
  - Multiple Risk constraints $x^t Q x \leq c$
  - Multiple Tracking Error constraints $(x - b)^t Q (x - b)) \leq c$
  - Soft constraints/objectives
    - Add "slack" to the constraint $\sum_{i \in S} x_i = c$ *is modified to* $\sum_{i \in S} x_i + s = c$
    - Add s^2 **to the objective** as a penalty term

How do Practitioners "fix" MVO? (Extensions)

- **Transaction cost models**
  - Linear: (add to the objective $\sum_{i \in U} \gamma_i t_i$)
  - Piecewise Linear (convex)

- **Market impact models**
  - Quadratic (add to the objective $\sum_{i \in U} \gamma_i t_i^2$)
  - Piecewise Linear

- **Risk Models (Common factors)**
  - Exploit mathematical structure of factor models
  - Factor related constraints/objectives
    - Q matrix is split into specific and factor risk
    - $Q = E^T R E + S^2$
    - *E = Exposure matrix, R = Factor Covariance Matrix (dense), S = Specific risk matrix (diagonal)*
    - Typically 50-100 factors are used

How do Practitioners "fix" MVO? (Extensions)

- **Fixed Transaction Costs**
  if $t_i > 0$ then *add* c_i to the *objective* Fixed Costs
- **Threshold Transactions**
  $t_i = 0$ or $t_i \geq c$
- **Threshold Holdings**
  $x_i = 0$ or $x_i \geq c$
- **Maximum number of Transactions/Holdings**
  $|\{i \mid x_i \neq 0\}| \leq c$
- **Round Lots on Transactions**
  $t_i \in \{kc \mid k = 1,2,\ldots\}$
- **If … then … else conditions**
  if {*linear expression*} then {*linear expression*}
  else {*linear expression*}

Practical Solution of MVO with Extensions

- In its general form, the problem is a *Quadratic Objective, Quadratic and Linearly Constrained Mixed-Integer (Disjunctive) Programming Program*

- Even though this problem is “hard” we can “solve” efficiently most practical instances in a few seconds

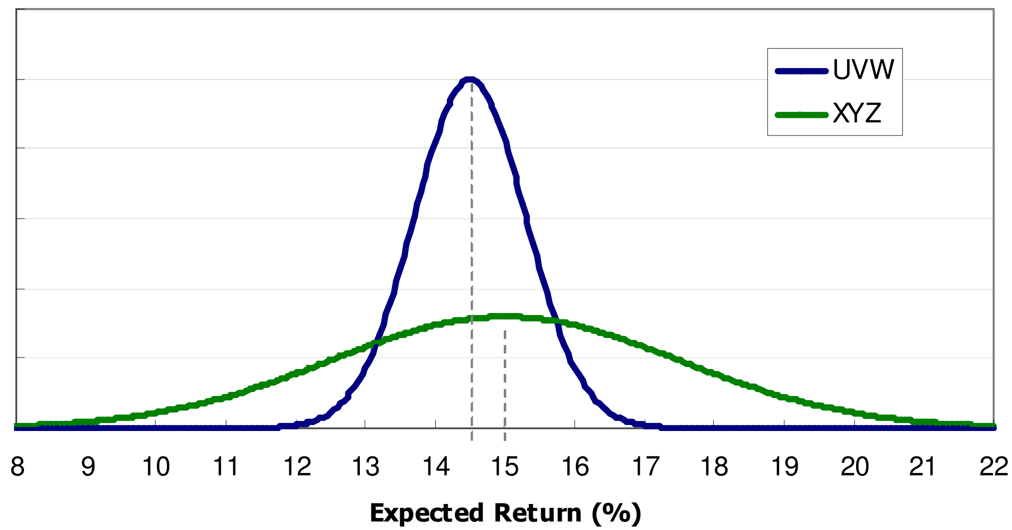
- Our algorithm includes
  - **Preprocessing**: Problem reduction and formulation improvement
  - **Strong relaxation:** Reformulation and strengthening techniques
  - **Heuristics**: Used to find feasible solutions (portfolios) fast
    - Relax-and-Fix
    - Diving
  - **Branching**: Specialized branching to exploit the structure
    - Cardinality constraints
    - Semi-continuous variables
    - Disjunctive statements

Is this Enough?

- Assume you are indifferent (from a risk perspective) between the two assets, how would you weigh these two assets in the optimal portfolio?

	Expected Return
XYZ	15.0%
UVW	14.5%

- Would you make the same choice if you knew the distribution of expected returns?



Improving Stability

Three Asset Example: shorting allowed, budget constraint

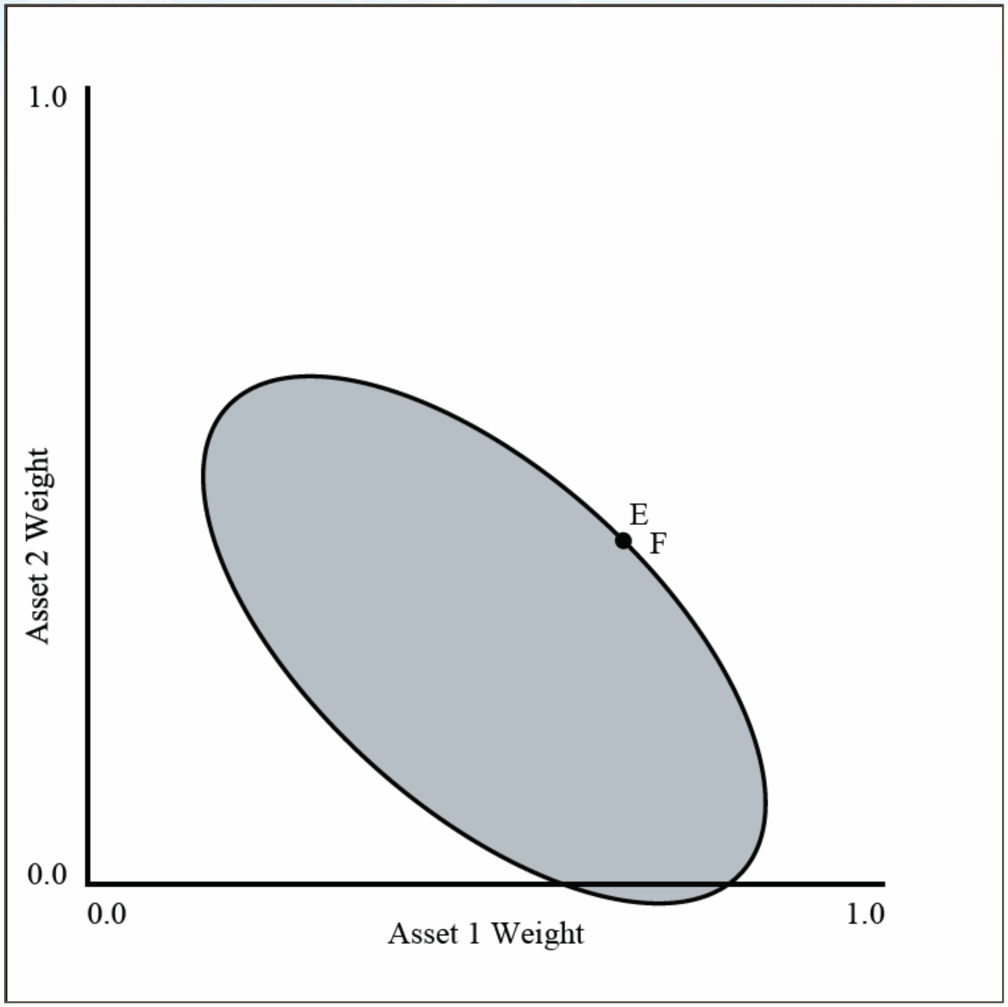
- Expected returns and standard deviations (correlations = 20%)

	α^1	α^2	σ
Asset 1	7.15%	7.16%	20%
Asset 2	7.16%	7.15%	24%
Asset 3	7.00%	7.00%	28%

- Optimal weights

	Portfolio E	Portfolio F
Asset 1	67.18%	67.26%
Asset 2	43.10%	43.01%
Asset 3	-10.28%	-10.28%

Graphical Representation



Improving Stability II

Three Asset Example: no shorting, budget constraint

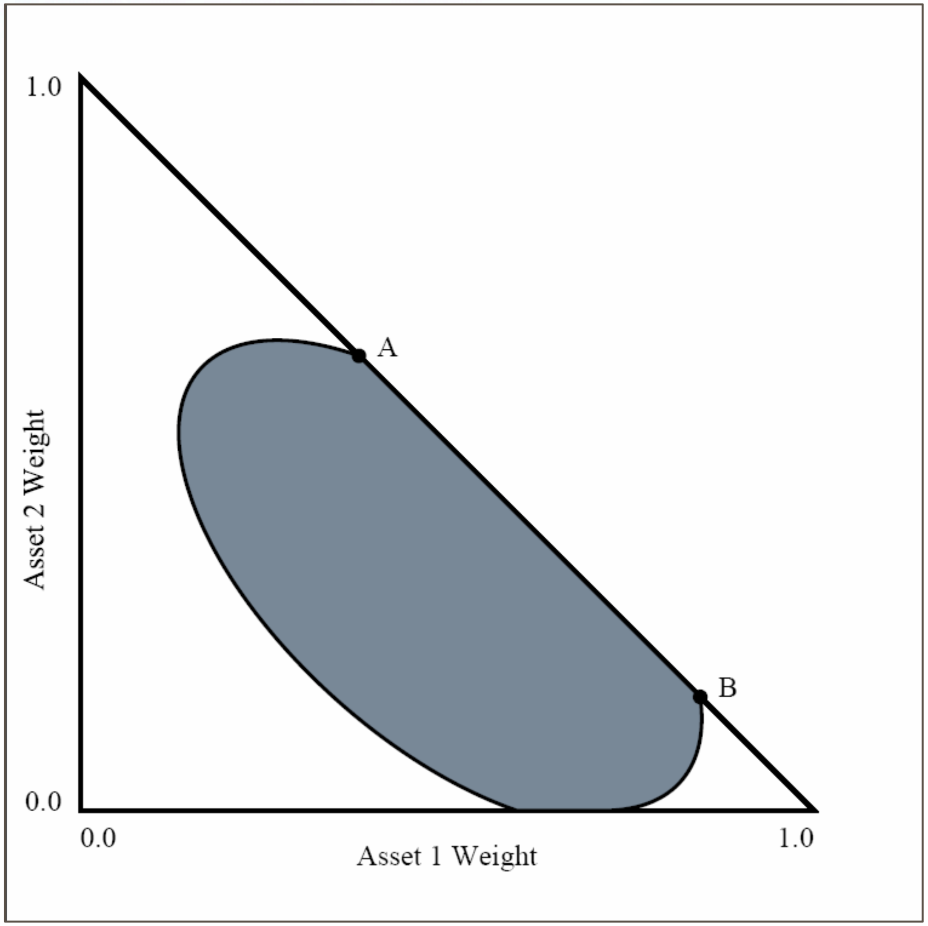
- Expected returns and standard deviations (correlations = 20%)

	α^1	α^2	σ
Asset 1	7.15%	7.16%	20%
Asset 2	7.16%	7.15%	24%
Asset 3	7.00%	7.00%	28%

- Optimal weights

	Portfolio A	Portfolio B
Asset 1	38.1%	84.3%
Asset 2	61.9%	15.7%
Asset 3	0.0%	0.0%

Constraints Creates Instability



Stability Experiment on the Dow 30

Instability due to changes in Expected Returns:

- Use expected returns and covariance from Idzorek (2002) for Dow 30
- Randomly generate 10,000 expected return estimate vectors from a normal distribution with mean equal to the expected return and std equal to 0.1% of the of the std of return of the corresponding asset
- Run 10,000 traditional MVO and record the weights of the resulting portfolios – Use a fixed risk aversion coefficient

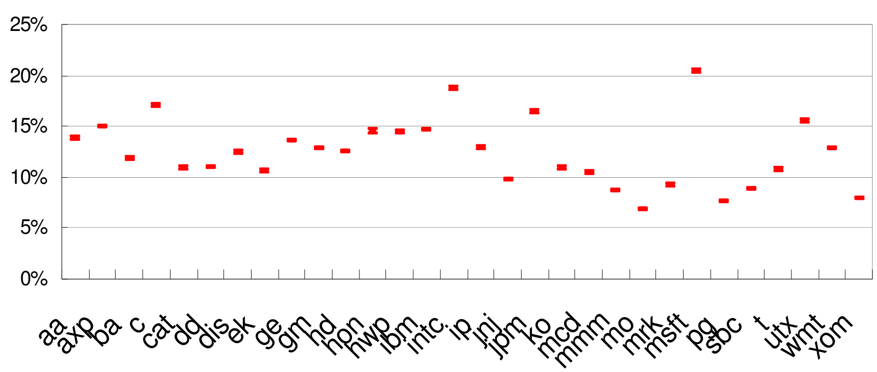


Figure 1: Range of Expected Returns used in MV Optimization

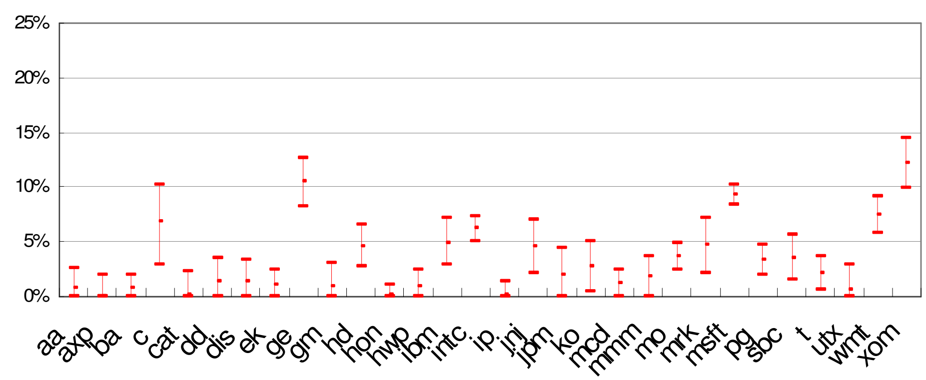


Figure 2: Range of Asset Weights for MV Optimization

Estimation Error Generates Inefficiencies

Estimated Frontier

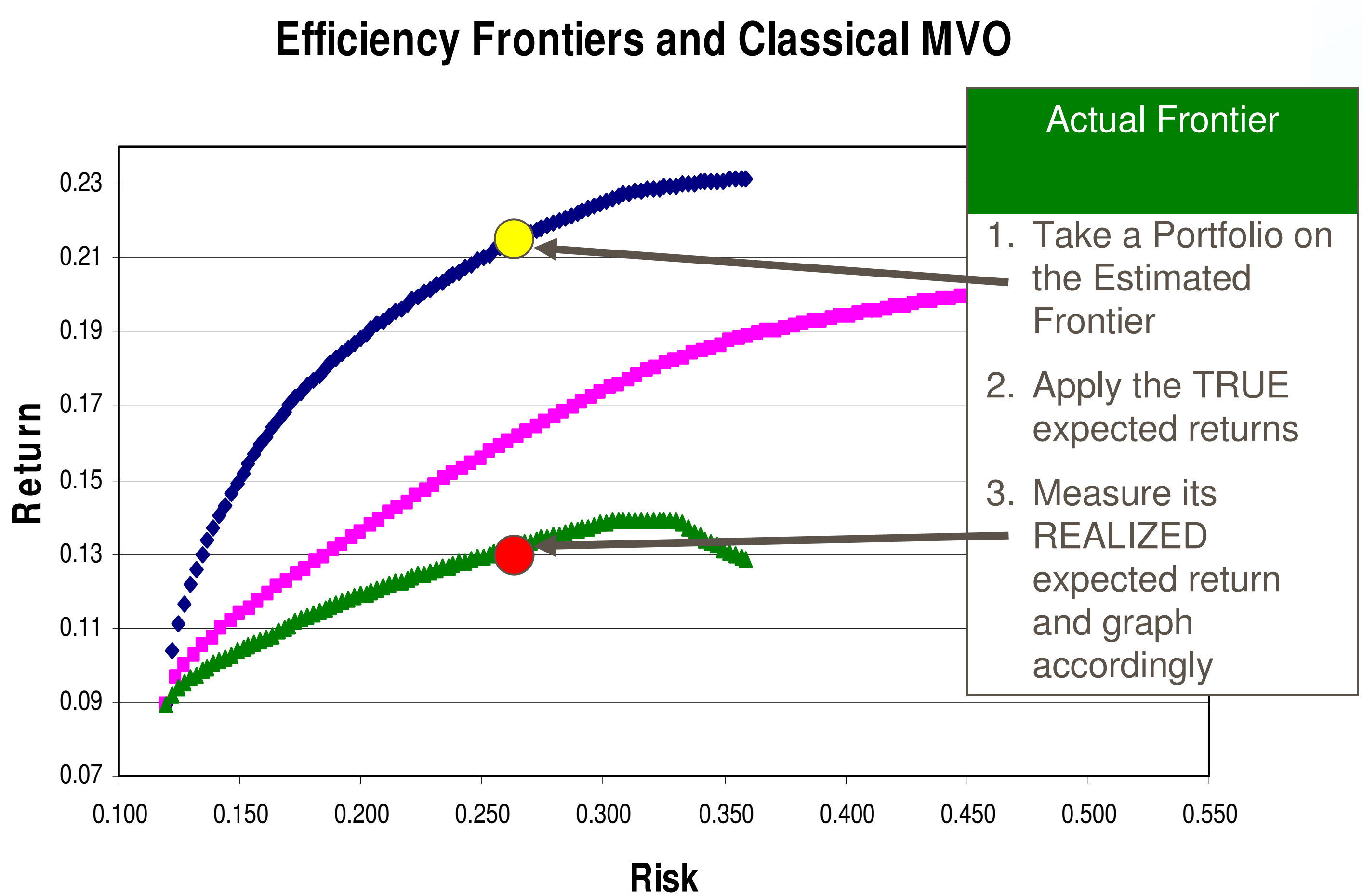
Efficiency Frontier computed using the estimated expected returns

True Frontier

Efficiency Frontier computed using the true expected returns

Actual Frontier

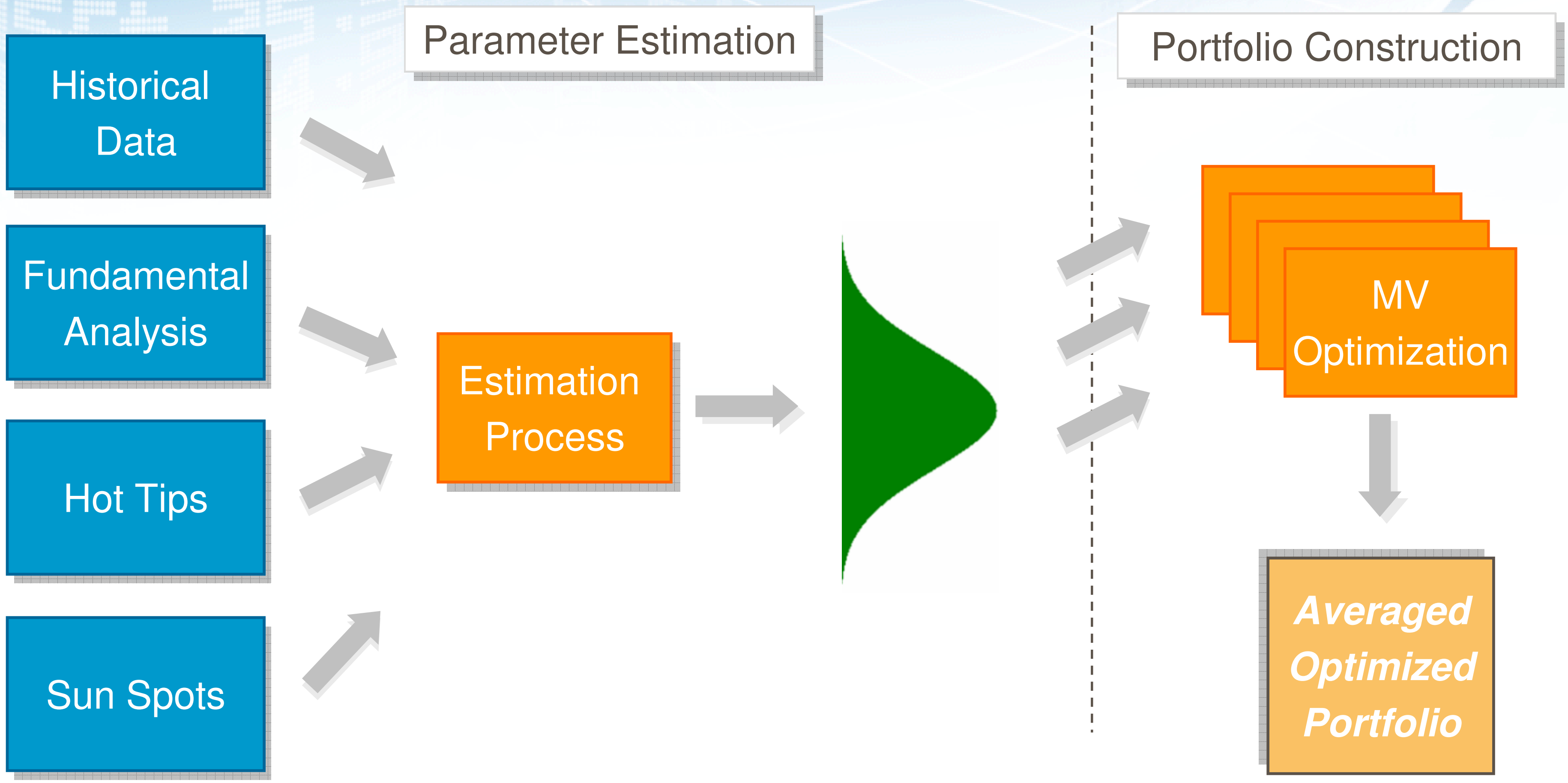
Return for the portfolios in the Estimated Frontier using the true expected returns



- How does the True Efficiency Frontier differ from the Actual Frontier?

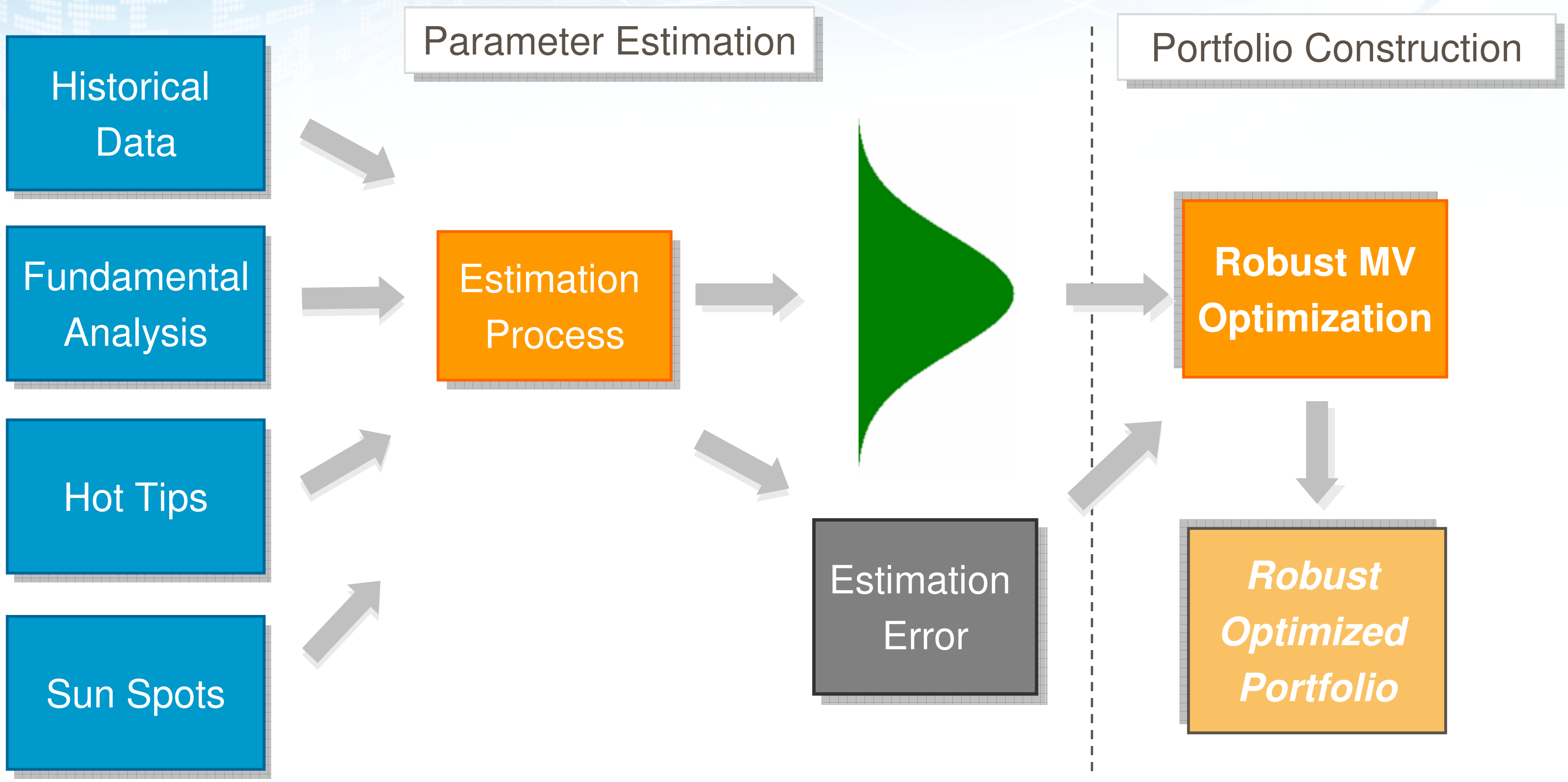
* See Broadie (1993) for a detailed discussion of estimated frontiers

One Possible Solution



- A byproduct of the estimation process is a distribution of estimated expected returns, and not a point forecast
- One option is to sample from the distribution and average the resulting portfolios: resampling

Integrating the Estimation Process and Robust MVO



- **Robust MVO uses explicitly the distribution of forecasted expected returns (estimation error) to find a robust portfolio in ONE step**

The Proposed Solution: Robust MVO

Robust Mean-Variance Optimization relies on “Robust Optimization” to solve the Portfolio Construction Problem

- What is Robust Optimization?
  - An optimization process that incorporates uncertainties of the inputs into a deterministic framework
  - It explicitly considers estimation error within the optimization process
  - It was developed independently by Ben-Tal and Nemirovski. Initial applications were in the area of engineering
- What are the advantages of using Robust MVO?
  - Recognize that there are errors in the estimation process and directly “exploit” that knowledge
  - Address practical portfolio construction constraints directly and explicitly
  - Solve the Robust MVO problem “efficiently” in “roughly” the same time as ONE classical mean variance optimization problem
- How do we solve Robust MVO problems?
  - The Robust MVO problem can be formulated as a “Second Order Cone Programming Problem” (Linear Programming over second-order cones)
  - Interior Point Algorithms are used to optimize SOCPs

> “When solving for the efficient portfolios, the differences in precision of the estimates should be explicitly incorporated into the analysis”
>
> H. Markowitz

Robust Optimization Background

- **Literature Review:** Even though Robust Optimization is relatively a new discipline, there is already an extensive literature in the subject for portfolio management
  - A. Ben-Tal and A.S. Nemirovski, "Robust convex optimization", Math. Operations Research, 1998
  - A. Ben-Tal and A.S. Nemirovski, "Robust solutions to uncertain linear programs", Operations Research Letters, 1999
  - L. El Ghaoui, F. Oustry, and H. Lebret, "Robust solutions to uncertain semidefinite programs", SIAM J. of Optimization, 1999
  - M. Lobo and S. Boyd, “The Worst Case Risk of a Portfolio”, 1999
  - R. Tütüncü and M. Koenig, “Robust Asset Allocation”, 2002
  - D. Goldfarb and G. Iyengar, “Robust Portfolio Selection Problems”, Math of OR, 2003
  - L. Garlappi, R. Uppal, and T. Wang, “Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach”, 2004
  - D. Bertsimas and M. Sim, Robust Discrete Optimization and Downside Risk Measures, 2005

Maximizing Robust Expected Returns

How do we maximize Robust Expected Returns?

- Assume the vector of expected returns $\alpha \sim N(\alpha^*,\Sigma)$
- Define an elliptical confidence region around the vector of estimated expected returns α^* as $\{\alpha: (\alpha - \alpha^*)^T\Sigma^{-1}(\alpha - \alpha^*) \leq k^2\}$

  (if the errors are normally distributed k^2 comes from the chi-squared distribution)
- **Robust Objective**: The optimization problem is defined as:

$$\textbf{Max (Min E(return))} = \mathbf{Max}_x\ \mathbf{Min}_{\alpha \in B(\alpha^*)}\ \mathbf{E}(\alpha x)$$

- And $B(\alpha^*)$ is the region around α^* that will be taken into account as potential errors in the estimates of expected returns

"A robust objective adjusts estimated expected returns to counter the (negative) effect that optimization has on the estimation errors that are present in the estimated expected returns"

Intuitive Derivation of Robust MVO

Estimated Frontier

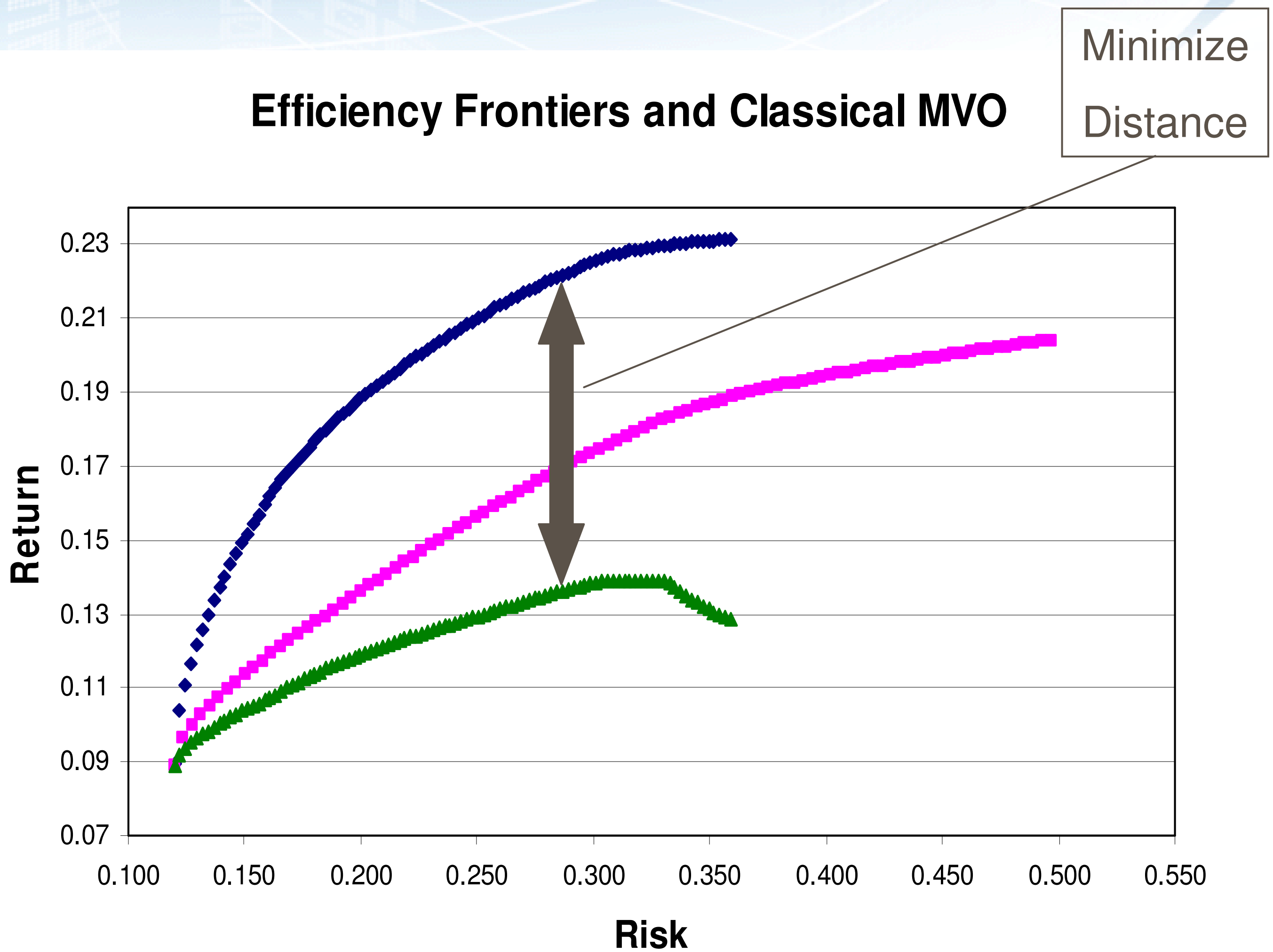
Efficiency Frontier computed using the estimated expected returns

True Frontier

Efficiency Frontier computed using the true expected returns

Actual Frontier

Return for the portfolios in the Estimated Frontier using the true expected returns



Mathematical Formulation of Robust MVO

Maximize expected return

Additional term that "corrects" for estimation error

Estimation Error "Aversion"

Additional term that "corrects" for risk

Risk "Aversion"

$$\left.\begin{aligned} &\max \alpha^{*} x - k\left\|\Sigma^{\frac{1}{2}} x\right\| - \lambda(x' Q x) \\ &st \quad e^{t} x = B \\ &\qquad x \geq 0 \end{aligned}\right\}$$

Second Order Cone Programming Problem

α^{*} vector of expected returns

Σ covariance matrix of estimated expected returns

Q covariance matrix of returns

k estimation error aversion

λ risk aversion

Impact of the Proposed Solution

- **Measuring the improvement: How do we know we are doing better?**
  - Reducing overestimation/underestimation
    - Compute the difference between the estimated and actual efficient frontiers
  - Improving stability
    - Compute a “measure” of the variability of weights given the variability in expected returns
  - Improving the “information transfer coefficient”, a measure that explains how information is being transferred
    - Measure of how efficiently an investment process is able to use the forecasting information it generates

Intuition behind Robust MVO: Simple Example

Three Asset Example: no shorting, budget constraint

- Expected returns and standard deviations (correlations = 20%)

	α^1 (A)	α^2 (B)	σ
Asset 1	7.15%	7.16%	20%
Asset 2	7.16%	7.15%	24%
Asset 3	7.00%	7.00%	28%

- Optimal weights

	High Aversion		Medium Aversion		Low Aversion	
	A	B	A	B	A	B
Asset 1	35.26%	35.68%	43.38%	45.54%	47.36%	52.65%
Asset 2	35.69%	35.27%	45.55%	43.39%	52.64%	47.35%
Asset 3	29.05%	29.05%	11.07%	11.07%	0.0%	0.0%

Reducing Overestimation/Underestimation

Estimated Robust Frontier

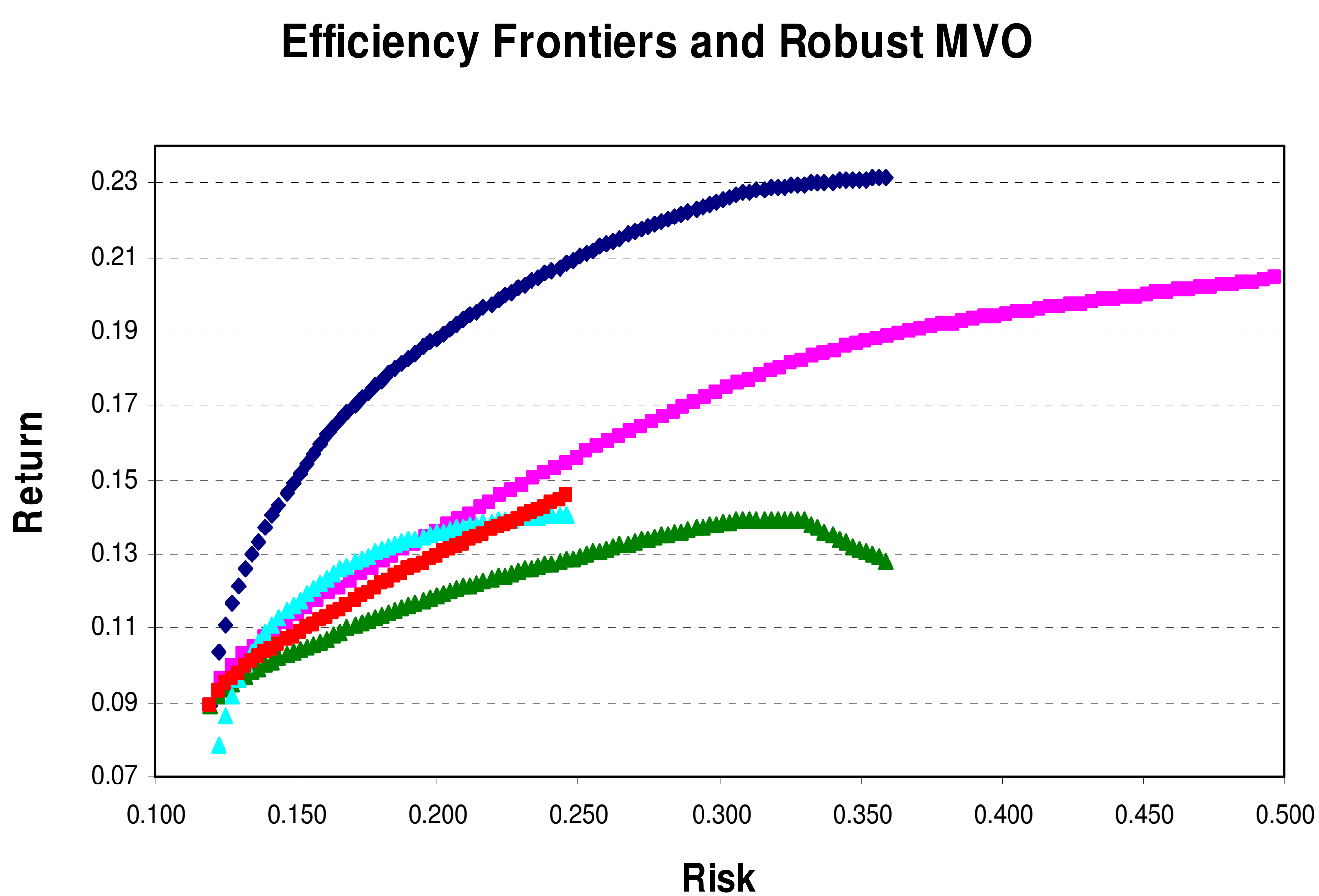
Robust Efficiency Frontier computed using the estimated expected returns and the estimation error

True Frontier

Efficiency Frontier computed using the true expected returns

Actual Robust Frontier

Return for the portfolios in the Estimated Frontier using the true expected returns



Improving Optimal Portfolio Stability and Intuition

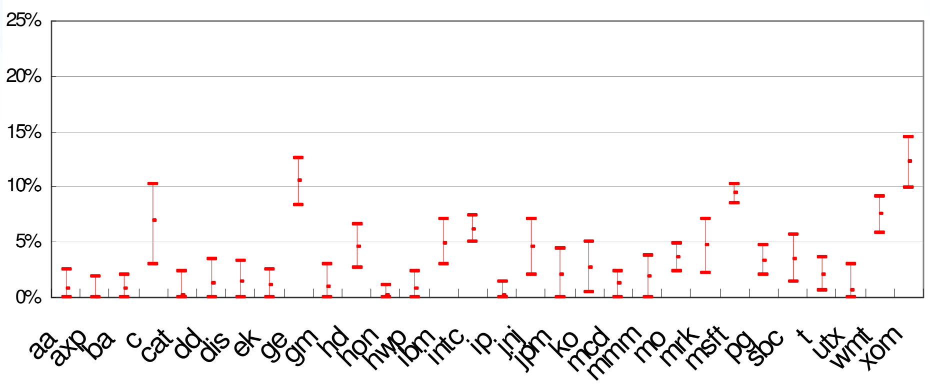


Figure 2: Range of Asset Weights for MV Optimization

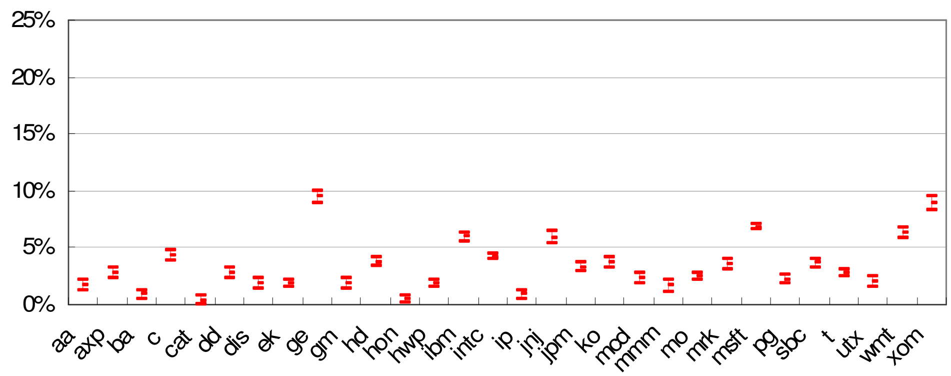
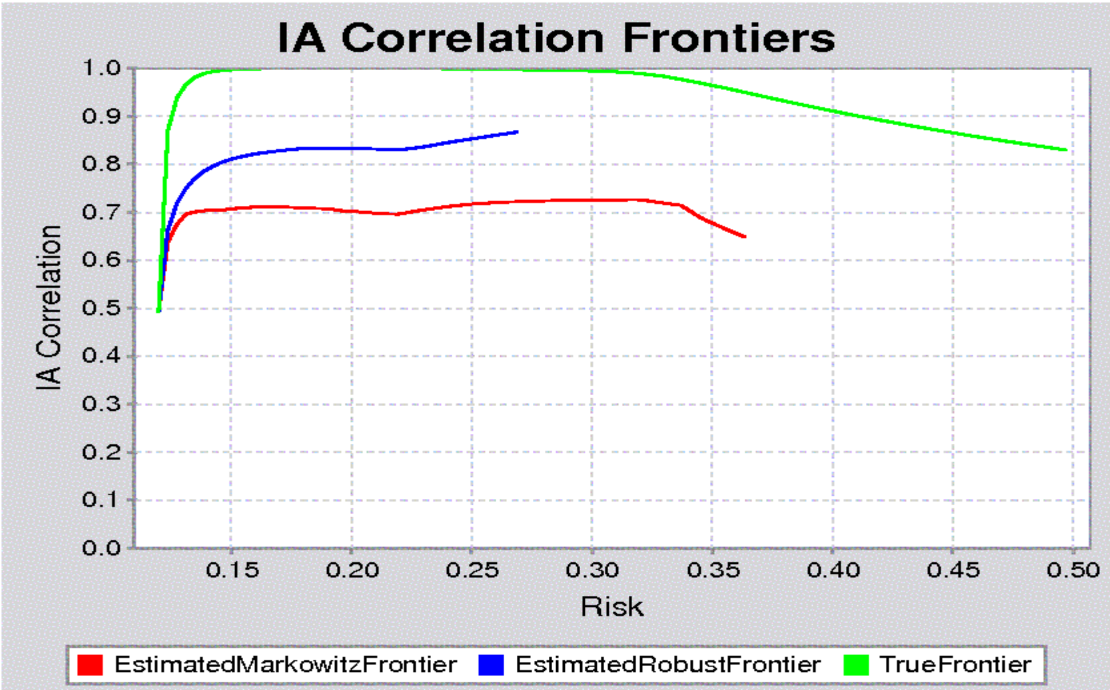


Figure 2: Range of Asset Weights for Robust Optimization

- Resultant asset weights using error maximized optimization vs. Robust MVO for the prior Dow 30 example
  - Lower ranges in asset weights
  - Less variability across asset weights

Improving the Transfer of Information

- Define a measure that allows us to determine how the information contained in the estimated expected returns is being "transferred" to the portfolio via the optimizer.
- We use the correlation of the implied alphas to the true alphas



Summary

- **Robust MVO** incorporates information about the estimation process directly into the optimization problem
- **Robust MVO** takes into account those estimation errors when computing the portfolio that maximizes utility
- **Robust MVO** improves performance through less trading and better use of the information at hand
- **Robust MVO** is a one-pass procedure which is efficiently implemented through an SOCP algorithm, it allows for the addition of other constraints
- **Robust MVO** naturally diversifies, even in the absence of a risk model

The End – Thank You

Improving Performance

- Start with a predefined covariance matrix and a vector of expected returns
- Randomly generate a time-series of returns for each asset, with the appropriate correlation
- To get estimated expected returns, we use "simplistic" estimators that average returns over previous periods with some weight assigned to current period (to put some look-ahead bias)
- To get the distribution of errors for estimated expected return we compute the covariance matrix over the same time periods. We scale the resulting matrix by a factor 1/v
- Sharpe Ratio is computed once, at the end of each run by taking the actual returns divided by their STD
- Each back-test is run 100 times
- For each time-period, we solve a problem with the following constraints
    - Risk constraint using the "true" covariance matrix (10% dollar-neutral)
    - Long/Only – Active No shorting
    - Turnover constraint at 7.5% (each way)
    - Asset bounds: 15% L/S [-15%,15%]

Simulated Back-Test Results (Dollar-Neutral)

		MVO		Robust MVO			
	Confidence Level	Ann. Return*	Sharpe Ratio*	Ann. Return*	Sharpe Ratio*	% Imp. Sharpe Ratio	% Improved Returns
Low Info	High	0.59%	0.1289	1.09%	0.1455	13%	73%
	Medium	0.59%	0.1289	1.95%	0.1767	37%	70%
	Low	0.59%	0.1289	2.95%	0.2260	75%	76%
	Very Low	0.59%	0.1289	3.77%	0.2812	118%	78%
High Info	High	1.49%	0.1657	2.02%	0.1847	11%	74%
	Medium	1.49%	0.1657	3.11%	0.2307	39%	72%
	Low	1.49%	0.1657	4.23%	0.2927	77%	80%
	Very Low	1.49%	0.1657	5.21%	0.3653	120%	81%

* Averages computed over 100 runs of the back-tests

Simulated Back-Test Results (Active Management 5%)

		MVO		Robust MVO			
	Confidence Level	Ann. Return*	Inf. Ratio*	Ann. Return*	Inf. Ratio*	% Imp. Inf. Ratio	% Improved Returns
Low Info	High	4.22%	0.5524	4.43%	0.5840	6%	77%
Low Info	Medium	4.22%	0.5524	4.80%	0.6379	15%	77%
Low Info	Low	4.22%	0.5524	5.00%	0.6641	20%	81%
Low Info	Very Low	4.22%	0.5524	5.02%	0.6607	20%	70%
High Info	High	6.45%	0.8022	6.72%	0.8465	6%	79%
High Info	Medium	6.45%	0.8022	7.17%	0.9234	15%	81%
High Info	Low	6.45%	0.8022	7.36%	0.9594	20%	80%
High Info	Very Low	6.45%	0.8022	7.33%	0.9555	19%	74%

* Averages computed over 100 runs of the back-tests