Robust Portfolio Construction

Presentation to
Workshop on Mixed Integer Programming

University of Miami

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Glossary

Assets (n)
Investable securities (U), typically stocks (equities)

Portfolio
Holdings: Initial (h), final (x) (holdings are represented in % or dollars)
Long holdings: (i : x_i > 0)
Short holdings: (i : x_i < 0)

Benchmark
A market portfolio: S&P 500, Russell 1000 (typically market-cap weighted) (b)

Budget
The total amount invested (B)

Expected Returns (Expected Active Returns)
A vector (α) of expectations of return (in percent), expected return of a portfolio α’x (α’(x-b))

Covariance of Returns
A matrix (Q) representing the forecasted covariances of returns

Predicted Risk of a portfolio
x’Qx

Predicted Tracking Error
(x-b)’Q(x-b)
Roadmap of the Quant Process

- Historical Data
- Fundamental Analysis
- Hot Tips
- Sun Spots

Parameter Estimation

- Estimation Process
- Forecasted Expected Returns
- Forecasted Risk

Portfolio Construction

- Business Rules
- MV Optimization
- Optimized Portfolio
Mathematical Models (MVO)

Active Management

\[ \text{Max } \alpha' x - \lambda (x - b)' Q (x - b) \]

st. \[ \sum_{i \in U} x_i = B \]

\[ x_i \geq 0, i \in U \]

- Expected Returns
- Risk Aversion
- Budget
- Tracking Error
- No Shorting
Why Don’t Practitioners Use MVO Extensively?

- Naïve portfolio rules, such as equal weighting, can outperform traditional MVO (Jobson and Korkie)

- Optimal portfolios from MVO are not necessarily well diversified (Jorion) or intuitive (Several authors)

- MV Optimizers have the “Error Maximization Property”. MVO will tend to overweight assets with positive estimation error and underweight assets with negative estimation error (Several authors)

- Unbiased risk and expected return estimators still lead to a biased estimate of the efficiency frontier (Several authors)

Portfolio Managers spend most of their time “cleaning up” the optimal portfolio provided by MVO
Criticisms of MVO

Literature Review: There is an extensive literature that studies the effects of estimation error in classical mean variance optimization

- Jobson and Korkie, “Putting Markowitz Theory to Work”, JPM, 1981 (and related work)
- Chopra and Ziemba, The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice”, JPM, 1993
How do Practitioners “fix” MVO? (Extensions)

**Simple (Linear):**
- Initial holdings \( h \)
- Transaction variables \( t = |x - h| \)
- Limits on holdings/trades \( x \leq u, t \leq v \)
- Industry/Sector Holdings \( \sum_{i \in S} x_i \leq c \)
- Active Holdings, Industry/Sector Active Holdings \( \sum_{i \in S} |x_i - b_i| \leq c \)
- Limits on Turnover, Trading, Buys/Sells \( \sum_{i \in S} t_i \leq c \)

**Complex (Linear-Quadratic):**
- Long/Short Portfolios (eliminate \( x \geq 0 \))
- Multiple Risk constraints \( x^t Q x \leq c \)
- Multiple Tracking Error constraints \( (x - b)^t Q (x - b) \leq c \)
- Soft constraints/objectives
  - Add “slack” to the constraint \( \sum_{i \in S} x_i = c \) is modified to \( \sum_{i \in S} x_i + s = c \)
  - Add \( s^2 \) to the objective as a penalty term
How do Practitioners “fix” MVO? (Extensions)

- **Transaction cost models**
  - Linear: (add to the objective $\sum_{i \in U} \gamma^i t^i$)
  - Piecewise Linear (convex)

- **Market impact models**
  - Quadratic (add to the objective $\sum_{i \in U} \gamma^i t^i$)
  - Piecewise Linear

- **Risk Models (Common factors)**
  - Exploit mathematical structure of factor models
  - Factor related constraints/objectives
    - $Q$ matrix is split into specific and factor risk
    - $Q = E^T R E + S^2$
    - $E = Exposure matrix, R = Factor Covariance Matrix (dense), S = Specific risk matrix (diagonal)$
    - Typically 50-100 factors are used
How do Practitioners “fix” MVO? (Extensions)

- Fixed Transaction Costs
  
  if $t_i > 0$ then add $c_i$ to the objective Fixed Costs

- Threshold Transactions
  
  $t_i = 0$ or $t_i \geq c$

- Threshold Holdings
  
  $x_i = 0$ or $x_i \geq c$

- Maximum number of Transactions/Holdings
  
  $\left| \{i \mid x_i \neq 0 \} \right| \leq c$

- Round Lots on Transactions
  
  $t_i \in \{kc \mid k = 1,2,\ldots\}$

- If … then … else conditions
  
  if \{linear expression\} then \{linear expression\}
  else \{linear expression\}
In its general form, the problem is a Quadratic Objective, Quadratic and Linearly Constrained Mixed-Integer (Disjunctive) Programming Program.

Even though this problem is “hard” we can “solve” efficiently most practical instances in a few seconds.

Our algorithm includes:
- **Preprocessing**: Problem reduction and formulation improvement
- **Strong relaxation**: Reformulation and strengthening techniques
- **Heuristics**: Used to find feasible solutions (portfolios) fast
  - Relax-and-Fix
  - Diving
- **Branching**: Specialized branching to exploit the structure
  - Cardinality constraints
  - Semi-continuous variables
  - Disjunctive statements
Is this Enough?

- Assume you are indifferent (from a risk perspective) between the two assets, how would you weigh these two assets in the optimal portfolio?

<table>
<thead>
<tr>
<th></th>
<th>Expected Return</th>
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<tbody>
<tr>
<td>XYZ</td>
<td>15.0%</td>
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<tr>
<td>UVW</td>
<td>14.5%</td>
</tr>
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</table>

- Would you make the same choice if you knew the distribution of expected returns?
Improving Stability

Three Asset Example: shorting allowed, budget constraint

Expected returns and standard deviations (correlations = 20%)

<table>
<thead>
<tr>
<th></th>
<th>$\alpha^1$</th>
<th>$\alpha^2$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>7.15%</td>
<td>7.16%</td>
<td>20%</td>
</tr>
<tr>
<td>Asset 2</td>
<td>7.16%</td>
<td>7.15%</td>
<td>24%</td>
</tr>
<tr>
<td>Asset 3</td>
<td>7.00%</td>
<td>7.00%</td>
<td>28%</td>
</tr>
</tbody>
</table>

Optimal weights

<table>
<thead>
<tr>
<th></th>
<th>Portfolio E</th>
<th>Portfolio F</th>
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</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>67.18%</td>
<td>67.26%</td>
</tr>
<tr>
<td>Asset 2</td>
<td>43.10%</td>
<td>43.01%</td>
</tr>
<tr>
<td>Asset 3</td>
<td>-10.28%</td>
<td>-10.28%</td>
</tr>
</tbody>
</table>
Graphical Representation
Three Asset Example: no shorting, budget constraint

- Expected returns and standard deviations (correlations = 20%)

<table>
<thead>
<tr>
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<th>(\sigma)</th>
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<td>7.16%</td>
<td>7.15%</td>
<td>24%</td>
</tr>
<tr>
<td>Asset 3</td>
<td>7.00%</td>
<td>7.00%</td>
<td>28%</td>
</tr>
</tbody>
</table>

- Optimal weights

<table>
<thead>
<tr>
<th></th>
<th>Portfolio A</th>
<th>Portfolio B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>38.1%</td>
<td>84.3%</td>
</tr>
<tr>
<td>Asset 2</td>
<td>61.9%</td>
<td>15.7%</td>
</tr>
<tr>
<td>Asset 3</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>
Constraints Creates Instability

![Graph showing the relationship between Asset 1 Weight and Asset 2 Weight, with a shaded region and points A and B.](image)
Stability Experiment on the Dow 30

Instability due to changes in Expected Returns:

- Use expected returns and covariance from Idzorek (2002) for Dow 30
- Randomly generate 10,000 expected return estimate vectors from a normal distribution with mean equal to the expected return and std equal to 0.1% of the std of return of the corresponding asset
- Run 10,000 traditional MVO and record the weights of the resulting portfolios – Use a fixed risk aversion coefficient

Figure 1: Range of Expected Returns used in MV Optimization

Figure 2: Range of Asset Weights for MV Optimization
Estimation Error Generates Inefficiencies

- **Estimated Frontier**: Efficiency Frontier computed using the estimated expected returns.
- **True Frontier**: Efficiency Frontier computed using the true expected returns.
- **Actual Frontier**: Return for the portfolios in the Estimated Frontier using the true expected returns.

**Efficiency Frontiers and Classical MVO**

How does the True Efficiency Frontier differ from the Actual Frontier?

* See Broadie (1993) for a detailed discussion of estimated frontiers.
One Possible Solution

- A byproduct of the estimation process is a distribution of estimated expected returns, and not a point forecast.
- One option is to sample from the distribution and average the resulting portfolios: resampling.
Robust MVO uses explicitly the distribution of forecasted expected returns (estimation error) to find a robust portfolio in ONE step.
Robust Mean-Variance Optimization relies on “Robust Optimization” to solve the Portfolio Construction Problem

What is Robust Optimization?
- An optimization process that incorporates uncertainties of the inputs into a deterministic framework
- It explicitly considers estimation error within the optimization process
- It was developed independently by Ben-Tal and Nemirovski. Initial applications were in the area of engineering

What are the advantages of using Robust MVO?
- Recognize that there are errors in the estimation process and directly “exploit” that knowledge
- Address practical portfolio construction constraints directly and explicitly
- Solve the Robust MVO problem “efficiently” in “roughly” the same time as ONE classical mean variance optimization problem

How do we solve Robust MVO problems?
- The Robust MVO problem can be formulated as a “Second Order Cone Programming Problem” (Linear Programming over second-order cones)
- Interior Point Algorithms are used to optimize SOCPs

“When solving for the efficient portfolios, the differences in precision of the estimates should be explicitly incorporated into the analysis”

H. Markowitz
Robust Optimization Background

**Literature Review:** Even though Robust Optimization is relatively a new discipline, there is already an extensive literature in the subject for portfolio management

- D. Bertsimas and M. Sim, Robust Discrete Optimization and Downside Risk Measures, 2005
Maximizing Robust Expected Returns

How do we maximize Robust Expected Returns?

- Assume the vector of expected returns $\alpha \sim N(\alpha^*, \Sigma)$
- Define an elliptical confidence region around the vector of estimated expected returns $\alpha^*$ as \{\(\alpha: (\alpha - \alpha^*)^T \Sigma^{-1} (\alpha - \alpha^*) \leq k^2\)\}
  (if the errors are normally distributed $k^2$ comes from the chi-squared distribution)

- **Robust Objective**: The optimization problem is defined as:
  \[
  \text{Max} \ (\text{Min } E(\text{return})) = \text{Max}_x \ \text{Min}_{\alpha \in B(\alpha^*)} E(\alpha x)
  \]

- And $B(\alpha^*)$ is the region around $\alpha^*$ that will be taken into account as potential errors in the estimates of expected returns

“A robust objective adjusts estimated expected returns to counter the (negative) effect that optimization has on the estimation errors that are present in the estimated expected returns”
Intuitive Derivation of Robust MVO

Estimated Frontier
- Efficiency Frontier computed using the estimated expected returns

True Frontier
- Efficiency Frontier computed using the true expected returns

Actual Frontier
- Return for the portfolios in the Estimated Frontier using the true expected returns

Efficiency Frontiers and Classical MVO

Minimize Distance

- Minimize the distance between the Estimated Frontier and the True Frontier

Graph showing the relationship between Risk and Return for different frontiers.
Mathematical Formulation of Robust MVO

Maximize expected return

Additional term that “corrects” for estimation error

Estimation Error “Aversion”

Additional term that “corrects” for risk

Risk “Aversion”

\[
\begin{align*}
\text{Maximize} & \quad \alpha^* \mathbf{x} - k \left\| \frac{1}{2} \Sigma \mathbf{x} \right\| - \lambda (x'Qx) \\
\text{subject to} & \quad e' \mathbf{x} = B \\
& \quad x \geq 0
\end{align*}
\]

\( \alpha^* \) vector of expected returns

\( \Sigma \) covariance matrix of estimated expected returns

\( Q \) covariance matrix of returns

\( k \) estimation error aversion

\( \lambda \) risk aversion

Second Order Cone Programming Problem
Impact of the Proposed Solution

Measuring the improvement: How do we know we are doing better?

- Reducing overestimation/underestimation
  - Compute the difference between the estimated and actual efficient frontiers

- Improving stability
  - Compute a “measure” of the variability of weights given the variability in expected returns

- Improving the “information transfer coefficient”, a measure that explains how information is being transferred
  - Measure of how efficiently an investment process is able to use the forecasting information it generates
Intuition behind Robust MVO: Simple Example

Three Asset Example: no shorting, budget constraint

Expected returns and standard deviations (correlations = 20%)

<table>
<thead>
<tr>
<th></th>
<th>$\alpha^1$ (A)</th>
<th>$\alpha^2$ (B)</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>7.15%</td>
<td>7.16%</td>
<td>20%</td>
</tr>
<tr>
<td>Asset 2</td>
<td>7.16%</td>
<td>7.15%</td>
<td>24%</td>
</tr>
<tr>
<td>Asset 3</td>
<td>7.00%</td>
<td>7.00%</td>
<td>28%</td>
</tr>
</tbody>
</table>

Optimal weights

<table>
<thead>
<tr>
<th></th>
<th>High Aversion</th>
<th>Medium Aversion</th>
<th>Low Aversion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>Asset 1</td>
<td>35.26%</td>
<td>35.68%</td>
<td>43.38%</td>
</tr>
<tr>
<td>Asset 2</td>
<td>35.69%</td>
<td>35.27%</td>
<td>45.55%</td>
</tr>
<tr>
<td>Asset 3</td>
<td>29.05%</td>
<td>29.05%</td>
<td>11.07%</td>
</tr>
</tbody>
</table>
Reducing Overestimation/Underestimation

**Estimated Robust Frontier**
Robust Efficiency Frontier computed using the estimated expected returns and the estimation error

**True Frontier**
Efficiency Frontier computed using the true expected returns

**Actual Robust Frontier**
Return for the portfolios in the Estimated Frontier using the true expected returns

**Efficiency Frontiers and Robust MVO**

- **Return** is plotted along the vertical axis ranging from 0.07 to 0.23.
- **Risk** is plotted along the horizontal axis ranging from 0.100 to 0.500.
- The graph shows four distinct lines representing different frontiers:
  - Estimated Robust Frontier
  - True Frontier
  - Actual Robust Frontier

The graph illustrates the relationship between risk and return, highlighting the impact of estimation error on efficient portfolio selection.
Improving Optimal Portfolio Stability and Intuition

Figure 2: Range of Asset Weights for MV Optimization

Figure 2: Range of Asset Weights for Robust Optimization

- Resultant asset weights using error maximized optimization vs. Robust MVO for the prior Dow 30 example
  - Lower ranges in asset weights
  - Less variability across asset weights
Improving the Transfer of Information

- Define a measure that allows us to determine how the information contained in the estimated expected returns is being “transferred” to the portfolio via the optimizer.
- We use the correlation of the implied alphas to the true alphas.
Summary

- **Robust MVO** incorporates information about the estimation process directly into the optimization problem.

- **Robust MVO** takes into account those estimation errors when computing the portfolio that maximizes utility.

- **Robust MVO** improves performance through less trading and better use of the information at hand.

- **Robust MVO** is a one-pass procedure which is efficiently implemented through an SOCP algorithm, it allows for the addition of other constraints.

- **Robust MVO** naturally diversifies, even in the absence of a risk model.
The End – Thank You
Improving Performance

- Start with a predefined covariance matrix and a vector of expected returns
- Randomly generate a time-series of returns for each asset, with the appropriate correlation
- To get estimated expected returns, we use “simplistic” estimators that average returns over previous periods with some weight assigned to current period (to put some look-ahead bias)
- To get the distribution of errors for estimated expected return we compute the covariance matrix over the same time periods. We scale the resulting matrix by a factor $1/v$
- Sharpe Ratio is computed once, at the end of each run by taking the actual returns divided by their STD
- Each back-test is run 100 times
- For each time-period, we solve a problem with the following constraints
  - Risk constraint using the “true” covariance matrix (10% dollar-neutral)
  - Long/Only – Active No shorting
  - Turnover constraint at 7.5% (each way)
  - Asset bounds: 15% L/S [-15%,15%]
## Simulated Back-Test Results (Dollar-Neutral)

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>MVO Ann. Return*</th>
<th>MVO Sharpe Ratio*</th>
<th>Robust MVO Ann. Return*</th>
<th>Robust MVO Sharpe Ratio*</th>
<th>% Imp. Sharpe Ratio</th>
<th>% Improved Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Info</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>0.59%</td>
<td>0.1289</td>
<td>1.09%</td>
<td>0.1455</td>
<td>13%</td>
<td>73%</td>
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<tr>
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<td>0.1289</td>
<td>1.95%</td>
<td>0.1767</td>
<td>37%</td>
<td>70%</td>
</tr>
<tr>
<td>Low</td>
<td>0.59%</td>
<td>0.1289</td>
<td>2.95%</td>
<td>0.2260</td>
<td>75%</td>
<td>76%</td>
</tr>
<tr>
<td>Very Low</td>
<td>0.59%</td>
<td>0.1289</td>
<td>3.77%</td>
<td>0.2812</td>
<td>118%</td>
<td>78%</td>
</tr>
<tr>
<td>High Info</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>1.49%</td>
<td>0.1657</td>
<td>2.02%</td>
<td>0.1847</td>
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<td>0.1657</td>
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<td>80%</td>
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<tr>
<td>Very Low</td>
<td>1.49%</td>
<td>0.1657</td>
<td>5.21%</td>
<td>0.3653</td>
<td>120%</td>
<td>81%</td>
</tr>
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</table>

* Averages computed over 100 runs of the back-tests

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## Simulated Back-Test Results (Active Management 5%)

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>MVO</th>
<th>Robust MVO</th>
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</thead>
<tbody>
<tr>
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<td>4.22%</td>
<td>0.5524</td>
</tr>
<tr>
<td>Medium</td>
<td>4.22%</td>
<td>0.5524</td>
</tr>
<tr>
<td>Low</td>
<td>4.22%</td>
<td>0.5524</td>
</tr>
<tr>
<td>Very Low</td>
<td>4.22%</td>
<td>0.5524</td>
</tr>
<tr>
<td>High</td>
<td>6.45%</td>
<td>0.8022</td>
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<tr>
<td>Medium</td>
<td>6.45%</td>
<td>0.8022</td>
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<tr>
<td>Low</td>
<td>6.45%</td>
<td>0.8022</td>
</tr>
<tr>
<td>Very Low</td>
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<td>0.8022</td>
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</table>

*Averages computed over 100 runs of the back-tests*