On connections between mixed-integer rounding and superadditive lifting

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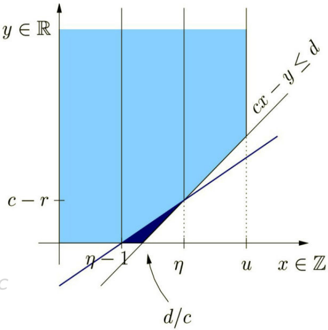
Joint work with Oktay Günlük

Mixed Integer Rounding

$cx \leq d + y \quad (c > 0)$

$x \in \mathbb{Z}, y \in \mathbb{R}_+$



$\eta = \lceil d/c \rceil$ and $r = d - \lfloor d/c \rfloor c$

MIR inequality $(c - r)x - y \leq (c - r)(\eta - 1) = d - \eta r$ cuts off $(d/c, 0)$

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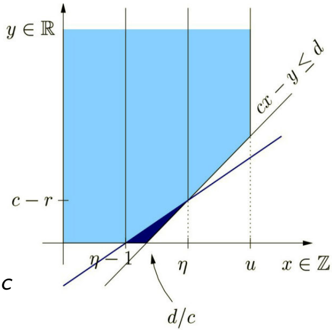

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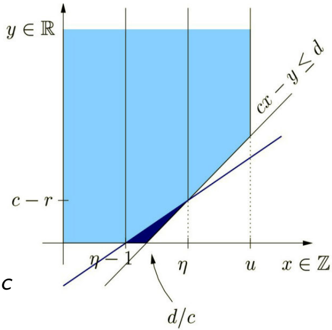

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$$\sum_{i \in I} a_i x_i \le b + t,\ \ x, t \ge 0$$

MIR:

$$\sum_{i \in I} [(1-f)\lfloor a_i \rfloor + (f_i - f)^+] x_i \le (1-f)\lfloor b \rfloor + t = b - \lceil b \rceil f + t$$

$$f_i = a_i - \lfloor a_i \rfloor, \quad f = b - \lfloor b \rfloor$$

Scale $1/\alpha$ & MIR:

$$\sum_{i \in I} [(\alpha - r)\lfloor a_i/\alpha \rfloor + (r_i - r)^+] x_i \le (\alpha - r)(\eta - 1) + t = b - \eta r + t$$

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$$s + \sum_{i \in I} a_i x_i \geq b, \;\; x, s \geq 0$$

MIR:

$$s + \sum_{i \in I} [f \lceil a_i \rceil - (f - f_i)^+] x_i \geq \lceil b \rceil f$$

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Superadditive Lifting for MIP

$$P(b) = \{x : Ax \leq b\}, \; x = (x_R, x_L, x_U), \; x_L \in \mathbb{Z}^L, \, x_U \in \mathbb{Z}^U$$

$$d = b - A_L l_L - A_U u_U$$

Consider $P_R(d) = \{x_R : A_R x_R \leq d\}$ and a tight valid inequality

$\pi_R x_R \leq \pi_o$ for $P_R(d)$

Value function: $v(h) = \max\{\pi_R x_R : x_R \in P_R(h)\}.$

Lifting function: $\Phi(a) = \pi_o - v(d-a)$

Theorem (A'01): If ϕ is superadditive and $\phi \leq \Phi$ then

$$\pi_R x_R + \sum_{i \in L} \phi(A_i)(x_i - l_i) + \sum_{i \in U} \phi(-A_i)(u_i - x_i) \leq \pi_o$$

is valid for $P(b)$. It is facet–defining for $conv(P(b))$ if $\phi = \Phi$ and $\pi_R x_R \leq \pi_o$ is facet–defining for $conv(P_R(d))$.
(Wolsey'78 - 0/1, Gu et al.'00: Mixed 0/1)

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Lifting function of MIR inequality $(c-r)x - y \leq d - \eta r$

$$\Phi(a) = d - \eta r - \max\{(c-r)x - y : cx - y \leq d - a,\ l \leq x \leq u,\ x \in \mathbb{Z}_+, y \in \mathbb{R}_+\}$$

$$\Phi(a) = \begin{cases} (\eta - u - 1)(c - r) & \text{if } a < d - uc \\ k(c-r) & \text{if } kc \leq a < kc + r \\ a - (k+1)r & \text{if } kc + r \leq a < (k+1)c \\ a - (\eta - l)r & \text{if } a \geq d - lc \end{cases} \qquad k \in \mathbb{Z}$$

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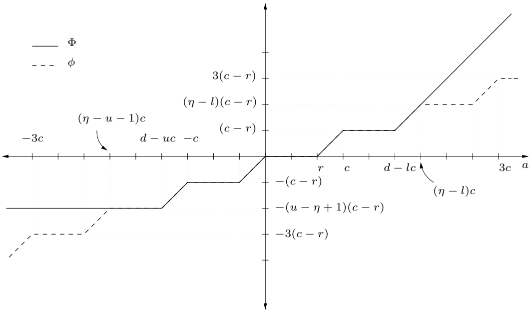

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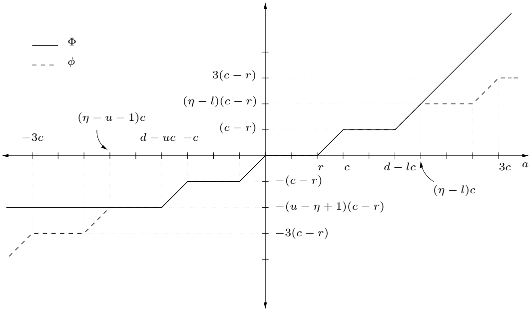

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$$(c-r)x - y \leq d - \eta r \quad \rightarrow \quad (c-r)x + \sum_{i \in I : a_i > 0} \Phi(a_i) x_i \leq d - \eta r$$

$$\Omega(a) = d - \eta r - \max\{(c-r)x + \sum_{i \in I : a_i > 0} \Phi(a_i) x_i : cx + \sum_{i \in I : a_i > 0} a_i x_i - y \leq d - a\}$$

$$= \begin{cases} u_{ih} \eta r + a & \text{if } m_{ih} - \delta_i \leq a \leq m_{ih}, \\ (u_{ih}\eta + k)r + a & \text{if } m_{ih} - \delta_i - (k+1)a_\ell + r \leq a \leq m_{ih} - \delta_i - k a_\ell, \\ u_{ih}\eta r + m_{ih} - \delta_i - (k+1)(a_\ell - r) & \text{if } m_{ih} - \delta_i - (k+1)a_\ell \leq a \leq m_{ih} - \delta_i - (k+1)a_\ell + r, \\ u_{su_S}\eta r + pr + a & \text{if } m_{J^{++}} - (p+1)a_\ell + r \leq a \leq m_{J^{++}} - p a_\ell, \\ m_{J^{++}} + u_{su_S}\eta r - (p+1)(a_\ell - r) & \text{if } m_{J^{++}} - (p+1)a_\ell \leq a \leq m_{J^{++}} - (p+1)a_\ell + r, \end{cases}$$

where $k \in \{0, 1, \ldots, \eta - 1\}$ and $p \in \mathbb{Z}_+$

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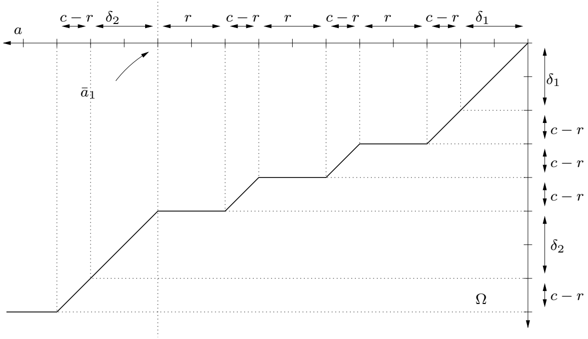
a
$c-r$
δ_2
r
$c-r$
r
$c-r$
r
$c-r$
δ_1
$\bar{a}_1$
δ_1
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Ω

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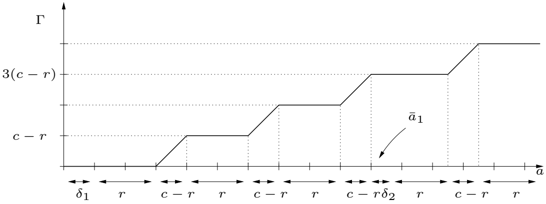
Γ
3(c − r)
c − r
$\bar{a}_1$
a
δ_1
r
c − r
r
c − r
r
c − r
δ_2
r
c − r
r

Experiments with lower-bounded variables

	CPLEX7.5				MIR cuts				Lifted cuts			
$\|I\| : \|C\| : \|R\|$	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time
250:1:50	111	67	68611	42	99	79	4416	4	101	83	1612	2
500:1:50	50	47	83061	63	99	77	6777	8	101	80	2929	4
250:1:75	177	56	196053	98	147	76	199883	183	153	79	40014	32
500:1:75	178	55	113745	92	147	75	176551	221	153	77	69242	79
250:1:100	220	71	251725	184	170	74	500125	417	204	75	226633	166
500:1:100	225	71	208047	283	196	73	256791	391	204	76	134211	187

Experiments with lower-and-upper bounded variables

$\lvert I \rvert : \lvert C \rvert : \lvert R \rvert$	CPLEX7.5				MIR cuts				Lifted cuts			
	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time
250:5:50	95	68	151252	63	213	79	7538	7	199	84	1070	2
500:5:50	50	42	128830	76	201	78	5045	7	206	83	1359	4
250:5:75	121	60	293392	144	408	77	86952	114	389	81	34412	38
500:5:75	122	59	207083	148	467	76	107139	213	440	80	17009	33
250:5:100	183	65	334504	203	691	80	179059	418	588	84	19257	29
500:5:100	130	68	305987	280	664	77	174394	446	613	81	33292	78
250:10:50	100	68	142438	67	245	77	15892	15	228	82	2443	3
500:10:50	50	38	166758	111	198	74	43397	46	235	80	2696	6
250:10:75	130	63	615537	370	474	75	213042	370	406	78	27591	38
500:10:75	105	61	228484	182	452	73	238559	519	477	78	33170	79
250:10:100	170	72	439032	361	777	75	301035	920	731	80	16689	55
500:10:100	136	72	302262	317	776	79	193303	489	678	81	18722	64
250:20:50	80	68	122693	73	228	76	186723	206	280	80	3247	6
500:20:50	48	45	131422	75	241	72	79649	101	252	79	3895	9
250:20:75	115	69	283218	176	416	73	424453	769	418	79	12852	22
500:20:75	118	69	390818	361	495	70	355781	779	479	78	20645	60
250:20:100	146	74	642986	531	849	70	383760	1594	656	79	26496	93
500:20:100	122	70	340536	434	702	71	305311	972	632	80	23930	88

" $\geq$ " *form* $\quad s + \sum_{i \in I} a_i x_i \geq b$

Scale by $1/\alpha$ & MIR: $\quad s + \sum_{i \in I} \mu_\alpha(a_i) x_i \geq \mu_\alpha(b)$

where $\mu_\alpha(a_i) = r\lceil a_i/\alpha \rceil - (r - r_i)^+$ *and* $r = b - \alpha\lfloor b/\alpha \rfloor$

" $\leq$ " *form* $\quad \sum_{i \in I} a_i x_i \leq b + t$

Scale by $1/\alpha$ & MIR: $\quad \sum_{i \in I} \phi_\alpha(a_i) x_i \leq \phi_\alpha(b) + t$

where $\phi_\alpha(a_i) = (\alpha - r)\lfloor a_i/\alpha \rfloor - (r_i - r)^+$

" ≥ " *form* $s + \sum_{i \in I} a_i x_i \geq b$

Scale by $1/\alpha$ & MIR: $s + \sum_{i \in I} \mu_\alpha(a_i) x_i \geq \mu_\alpha(b)$

where $\mu_\alpha(a_i) = r\lceil a_i/\alpha \rceil - (r - r_i)^+$ *and* $r = b - \alpha \lfloor b/\alpha \rfloor$

" ≤ " *form* $\sum_{i \in I} a_i x_i \leq b + t$

Scale by $1/\alpha$ & MIR: $\sum_{i \in I} \phi_\alpha(a_i) x_i \leq \phi_\alpha(b) + t$

where $\phi_\alpha(a_i) = (\alpha - r)\lfloor a_i/\alpha \rfloor - (r_i - r)^+$

MIR Lifting

Complement, scale & MIR (Marchand & Wolsey '01)

$$\sum_{i\in I} a_i x_i \leq b + s,\ x \in \{0,1\}^I,\ s \geq 0$$

For cover $C \subseteq I$ ie $\lambda := \sum_{i\in C} a_i - b > 0$, rewrite

$$s + \sum_{i\in C} a_i(1-x_i) + \sum_{i\in I\setminus C} -a_i x_i \geq \lambda$$

Scale $1/(\bar{a} := max_{i\in C} a_i)$ and MIR:

$$s + \sum_{i\in C} \min\{\lambda, a_i\}(1-x_i) + \sum_{i\in I\setminus C} \mu_{\bar{a}}(-a_i)x_i \geq \lambda$$

Rewrite

$$\sum_{i\in C} \min\{\lambda, a_i\}(x_i - 1) + \sum_{i\in I\setminus C} -\mu_{\bar{a}}(-a_i)x_i \leq -\lambda$$

For minimal cover C

$$\sum_{i\in C} \lambda x_i + \sum_{i\in I\setminus C} -\mu_{\bar{a}}(-a_i)x_i \leq \lambda(|C|-1)$$

MIR Lifting

Complement, scale & MIR (Marchand & Wolsey '01)

$$\sum_{i \in I} a_i x_i \leq b + s, \; x \in \{0,1\}^I, \; s \geq 0$$

For cover $C \subseteq I$ ie $\lambda := \sum_{i \in C} a_i - b > 0$, rewrite

$$s + \sum_{i \in C} a_i(1 - x_i) + \sum_{i \in I \setminus C} -a_i x_i \geq \lambda$$

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Rewrite

$$\sum_{i \in C} \min\{\lambda, a_i\}(x_i - 1) + \sum_{i \in I \setminus C} -\mu_{\bar{a}}(-a_i) x_i \leq -\lambda$$

For minimal cover C

$$\sum_{i \in C} \lambda x_i + \sum_{i \in I \setminus C} -\mu_{\bar{a}}(-a_i) x_i \leq \lambda(|C| - 1)$$

MIR Lifting

Complement, scale & MIR (Marchand & Wolsey '01)

$$\sum_{i \in I} a_i x_i \leq b + s, \;\; x \in \{0,1\}^I, \;\; s \geq 0$$

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Rewrite

$$\sum_{i \in C} \min\{\lambda, a_i\}(x_i - 1) + \sum_{i \in I \setminus C} -\mu_{\bar{a}}(-a_i) x_i \leq -\lambda$$

For minimal cover C

$$\sum_{i \in C} \lambda x_i + \sum_{i \in I \setminus C} -\mu_{\bar{a}}(-a_i) x_i \leq \lambda(|C| - 1)$$

MIR Lifting

Complement, scale & MIR (Marchand & Wolsey '01)

$$\sum_{i \in I} a_i x_i \le b + s, \;\; x \in \{0,1\}^I, \;\; s \ge 0$$

For cover $C \subseteq I$ ie $\lambda := \sum_{i \in C} a_i - b > 0$, rewrite

$$s + \sum_{i \in C} a_i(1 - x_i) + \sum_{i \in I \setminus C} -a_i x_i \ge \lambda$$

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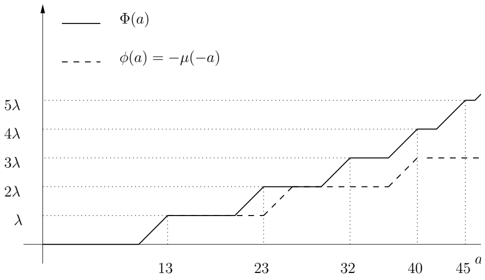
Rewrite

$$\sum_{i \in C} \min\{\lambda, a_i\}(x_i - 1) + \sum_{i \in I \setminus C} -\mu_{\bar{a}}(-a_i) x_i \le -\lambda$$

For minimal cover C

$$\sum_{i \in C} \lambda x_i + \sum_{i \in I \setminus C} -\mu_{\bar{a}}(-a_i) x_i \le \lambda(|C| - 1)$$

Example



$$13x_1 + 10x_2 + 9x_3 + 8x_4 + 5x_5 + ax_6 \leq 42 + s$$

$$C = \{1, 2, 3, 4, 5\}$$

$$\lambda = 13 + 10 + 9 + 8 + 5 - 42 = 3$$

Incorporating lower bound information into MIR

" $\geq$ " form

Assumption: $0 < a_1 < b < a_2$

$$s + a_1x_1 + a_2x_2 \geq b, \; s, x \geq 0$$

Scale $1/a_1$ & MIR:

$$s + rx_1 + r\lceil a_2/a_1 \rceil x_2 \geq r\lceil b/a_1 \rceil$$

where $r = b - a_1\lfloor b/a_1 \rfloor$

Coefficient improvement, scale $1/a_1$ & MIR:

$$s + a_1x_1 + bx_2 \geq b \quad (s, x, a_1 \geq 0)$$

$$s + rx_1 + r\lceil b/a_1 \rceil x_2 \geq r\lceil b/a_1 \rceil$$

Incorporating lower bound information into MIR

" $\geq$ " form

Assumption: $0 < a_1 < b < a_2$

$$s + a_1 x_1 + a_2 x_2 \geq b, \;\; s, x \geq 0$$

Scale $1/a_1$ & MIR:

$$s + r x_1 + r\lceil a_2/a_1 \rceil x_2 \geq r\lceil b/a_1 \rceil$$

where $r = b - a_1 \lfloor b/a_1 \rfloor$

Coefficient improvement, scale $1/a_1$ & MIR:

$$s + a_1 x_1 + b x_2 \geq b \quad (s, x, a_1 \geq 0)$$

$$s + r x_1 + r\lceil b/a_1 \rceil x_2 \geq r\lceil b/a_1 \rceil$$

Incorporating lower bound information into MIR

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$$s + a_1 x_1 + b x_2 \geq b \quad (s, x, a_1 \geq 0)$$

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where $r = b - a_1\lfloor b/a_1 \rfloor$

Coefficient improvement, scale $1/a_1$ & MIR:

$$s + a_1x_1 + bx_2 \geq b \quad (s, x, a_1 \geq 0)$$

$$s + rx_1 + r\lceil b/a_1 \rceil x_2 \geq r\lceil b/a_1 \rceil$$

In general: $(a_i \geq 0,\ i \in I)$

$$s + \sum_{i \in I} a_i x_i \geq b,\ s, x \geq 0$$

Scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I} \mu_\alpha(a_i) x_i \geq \mu_\alpha(b)$$

Coefficient improvement, scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{i \in I^+} b x_i \geq b$$

$$s + \sum_{i \in I \setminus I^+} \mu_\alpha(a_i) + \sum_{i \in I^+} \mu_\alpha(b) x_i \geq \mu_\alpha(b)$$

Same as lifted inequality (A'01) or two-step MIR inequality (Dash & Günlük'06)

$$s + \sum \Theta_\alpha(a_i) x_i \geq \Theta_\alpha(b)$$

In general: $(a_i \geq 0,\ i \in I)$

$$s + \sum_{i \in I} a_i x_i \geq b,\ \ s, x \geq 0$$

Scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I} \mu_\alpha(a_i) x_i \geq \mu_\alpha(b)$$

Coefficient improvement, scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{i \in I^+} b x_i \geq b$$

$$s + \sum_{i \in I \setminus I^+} \mu_\alpha(a_i) + \sum_{i \in I^+} \mu_\alpha(b) x_i \geq \mu_\alpha(b)$$

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Scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I} \mu_\alpha(a_i) x_i \geq \mu_\alpha(b)$$

Coefficient improvement, scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{i \in I^+} b x_i \geq b$$

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Scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I} \mu_\alpha(a_i) x_i \geq \mu_\alpha(b)$$

Coefficient improvement, scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{i \in I^+} b x_i \geq b$$

$$s + \sum_{i \in I \setminus I^+} \mu_\alpha(a_i) + \sum_{i \in I^+} \mu_\alpha(b) x_i \geq \mu_\alpha(b)$$

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In general: $(a_i \geq 0, \; i \in I)$

$$s + \sum_{i \in I} a_i x_i \geq b, \; s, x \geq 0$$

Scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I} \mu_\alpha(a_i) x_i \geq \mu_\alpha(b)$$

Coefficient improvement, scale $1/\alpha$ & MIR:

$$s + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{i \in I^+} b x_i \geq b$$

$$s + \sum_{i \in I \setminus I^+} \mu_\alpha(a_i) + \sum_{i \in I^+} \mu_\alpha(b) x_i \geq \mu_\alpha(b)$$

Same as lifted inequality (A'01) or two-step MIR inequality (Dash & Günlük'06)

$$s + \sum \Theta_\alpha(a_i) x_i \geq \Theta_\alpha(b)$$

" $\leq$ " form

Assumption : $0 < a_1 < b < a_2$

$$a_1x_1 + a_2x_2 \leq b + t, \;\; t, x \geq 0$$

Scale $1/a_1$ & MIR:

$$(a_1 - r)x_1 + (a_2 - r\lceil a_2/a_1\rceil)x_2 \leq b - r\lceil b/a_1\rceil + t$$

Coefficient improvement, scale $1/a_1$ & MIR:

$$(a_1 - r)x_1 + (a_2 - r\lceil b/a_1\rceil)x_2 \leq b - r\lceil b/a_1\rceil + t$$

In general:

$$\sum_{i \in I \setminus I^+} \phi_\alpha(a_i) + \sum_{i \in I^+}[a_i - b + \phi_\alpha(b)]x_i \leq \phi_\alpha(b) + t$$

Same as lifted inequality or two step MIR inequality

$$\sum_{i \in I} \Phi_\alpha(a_i)x_i \leq \Phi_\alpha(b) + t$$

" $\leq$ " form

Assumption : $0 < a_1 < b < a_2$

$$a_1 x_1 + a_2 x_2 \leq b + t, \;\; t, x \geq 0$$

Scale $1/a_1$ & MIR:

$$(a_1 - r)x_1 + (a_2 - r\lceil a_2/a_1 \rceil)x_2 \leq b - r\lceil b/a_1 \rceil + t$$

Coefficient improvement, scale $1/a_1$ & MIR:

$$(a_1 - r)x_1 + (a_2 - r\lceil b/a_1 \rceil)x_2 \leq b - r\lceil b/a_1 \rceil + t$$

In general:

$$\sum_{i \in I \setminus I^+} \phi_\alpha(a_i) + \sum_{i \in I^+} [a_i - b + \phi_\alpha(b)]x_i \leq \phi_\alpha(b) + t$$

Same as lifted inequality or two step MIR inequality

$$\sum_{i \in I} \Phi_\alpha(a_i)x_i \leq \Phi_\alpha(b) + t$$

" $\leq$ " form

Assumption : $0 < a_1 < b < a_2$

$$a_1 x_1 + a_2 x_2 \leq b + t, \ \ t, x \geq 0$$

Scale $1/a_1$ & MIR:

$$(a_1 - r)x_1 + (a_2 - r\lceil a_2/a_1 \rceil)x_2 \leq b - r\lceil b/a_1 \rceil + t$$

Coefficient improvement, scale $1/a_1$ & MIR:

$$(a_1 - r)x_1 + (a_2 - r\lceil b/a_1 \rceil)x_2 \leq b - r\lceil b/a_1 \rceil + t$$

In general:

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Same as lifted inequality or two step MIR inequality

$$\sum_{i \in I} \Phi_\alpha(a_i) x_i \leq \Phi_\alpha(b) + t$$

" $\leq$ " *form*

Assumption : $0 < a_1 < b < a_2$

$$a_1x_1 + a_2x_2 \leq b + t, \;\; t, x \geq 0$$

Scale $1/a_1$ & MIR:

$$(a_1 - r)x_1 + (a_2 - r\lceil a_2/a_1 \rceil)x_2 \leq b - r\lceil b/a_1 \rceil + t$$

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In general:

$$\sum_{i \in I \setminus I^+} \phi_\alpha(a_i) + \sum_{i \in I^+} [a_i - b + \phi_\alpha(b)]x_i \leq \phi_\alpha(b) + t$$

Same as lifted inequality or two step MIR inequality

$$\sum_{i \in I} \Phi_\alpha(a_i)x_i \leq \Phi_\alpha(b) + t$$

Incorporating lower and upper bound information into MIR

" $\geq$ " *form* Assumption : $a_1 < 0 < b < a_2$

$$s + a_1x_1 + a_2x_2 \geq b, \quad s, x \geq 0, \quad x_2 \leq u_2$$

Rewrite as

$$s + (a_1 + a_2u_2)x_1 + a_2(x_2 - u_2x_1) \geq b$$

Valid inequality

$$s + (a_1 + a_2u_2)x_1 + b(x_2 - u_2x_1) \geq b \quad (*)$$

$$s + (a_1 + (a_2 - b)u_2)x_1 + bx_2 \geq b$$

MIR ($\alpha = a_2$)

$$s + [b\lceil a_1/a_2\rceil - (b - r_1)^+]x_1 + bx_2 \geq b$$

$(*)$ dominates MIR if $u_2 < -a_1/a_2$

Incorporating lower and upper bound information into MIR

" $\geq$ " form Assumption : $a_1 < 0 < b < a_2$

$$s + a_1 x_1 + a_2 x_2 \geq b, \quad s, x \geq 0, \quad x_2 \leq u_2$$

Rewrite as

$$s + (a_1 + a_2 u_2) x_1 + a_2 (x_2 - u_2 x_1) \geq b$$

Valid inequality

$$s + (a_1 + a_2 u_2) x_1 + b(x_2 - u_2 x_1) \geq b \quad (*)$$

$$s + (a_1 + (a_2 - b) u_2) x_1 + b x_2 \geq b$$

MIR ($\alpha = a_2$)

$$s + [b \lceil a_1 / a_2 \rceil - (b - r_1)^+] x_1 + b x_2 \geq b$$

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$$s + a_1 x_1 + a_2 x_2 \geq b, \quad s, x \geq 0, \quad x_2 \leq u_2$$

Rewrite as

$$s + (a_1 + a_2 u_2) x_1 + a_2 (x_2 - u_2 x_1) \geq b$$

Valid inequality

$$s + (a_1 + a_2 u_2) x_1 + b (x_2 - u_2 x_1) \geq b \quad (*)$$

$$s + (a_1 + (a_2 - b) u_2) x_1 + b x_2 \geq b$$

MIR ($\alpha = a_2$)

$$s + [b \lceil a_1 / a_2 \rceil - (b - r_1)^+] x_1 + b x_2 \geq b$$

$(*)$ dominates MIR if $u_2 < -a_1 / a_2$

Incorporating lower and upper bound information into MIR

" $\geq$ " form Assumption : $a_1 < 0 < b < a_2$

$$s + a_1 x_1 + a_2 x_2 \geq b, \quad s, x \geq 0, \quad x_2 \leq u_2$$

Rewrite as

$$s + (a_1 + a_2 u_2) x_1 + a_2 (x_2 - u_2 x_1) \geq b$$

Valid inequality

$$s + (a_1 + a_2 u_2) x_1 + b(x_2 - u_2 x_1) \geq b \quad (*)$$

$$s + (a_1 + (a_2 - b) u_2) x_1 + b x_2 \geq b$$

MIR ($\alpha = a_2$)

$$s + [b \lceil a_1 / a_2 \rceil - (b - r_1)^+] x_1 + b x_2 \geq b$$

$(*)$ dominates MIR if $u_2 < -a_1 / a_2$

Let $k = \min\{u_2, \lceil -a_1/a_2 \rceil\}$

$$s + (a_1 + a_2 k)x_1 + a_2(x_2 - kx_1) \geq b$$

$$s + \min\{b, (a_1 + a_2 k)\}x_1 + b(x_2 - kx_1) \geq b \quad (**) \text{ is valid}$$

If $a_1 + a_2 k < b$, inequality

$$s/\alpha + (a_1 + a_2 k)x_1/\alpha + \lceil b/\alpha \rceil (x_2 - kx_1) \geq b/\alpha$$

is valid for any α s.t. $a_2/\alpha > \lceil b/\alpha \rceil$

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Equivalent to superadditive lifting or if $k < u_2$ to two-step MIR inequality

Let $k = \min\{u_2, \lceil -a_1/a_2 \rceil\}$

$$s + (a_1 + a_2 k)x_1 + a_2(x_2 - kx_1) \geq b$$

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" $\leq$ " form

Assumption : $a_1 < 0 < b < a_2$

$$a_1x_1 + a_2x_2 \leq b + t, \quad t, x \geq 0, \quad x_2 \leq u_2$$

$$s + a_1x_1 + a_2x_2 = b + t, \quad s, t, x \geq 0$$

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$$(a_1 - \mu_\alpha(a_1 + ka_2) + kr\lceil b/\alpha\rceil)x_1 + (a_2 - r\lceil b/\alpha\rceil)x_2 \leq b - r\lceil b/\alpha\rceil + t$$

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Let $I^+ := \{i \in I : a_i \geq b\} := \{1, \ldots, n\}$ be sorted in nonincreasing a_i.

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for any α st $\frac{\min_{i \in I^+} a_i}{\alpha} \geq \lceil b/\alpha \rceil$

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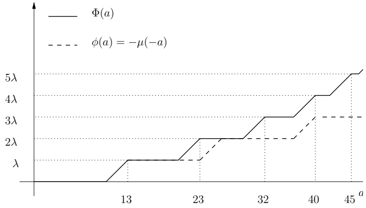

MIR lifting

$$\sum_{i=1}^{5} 3(1 - x_i) + \mu_{13}(-a)x_6 \geq 3$$

Say $I_6 = C$:

$$\sum_{i=1}^{5} a_i[(1 - x_i) - x_6] + (-ax_6 + \sum_{i=1}^{5} a_i) \geq 3$$

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Use $\alpha : \min_{i \in C} a_i/\alpha \geq \lceil \lambda/\alpha \rceil$

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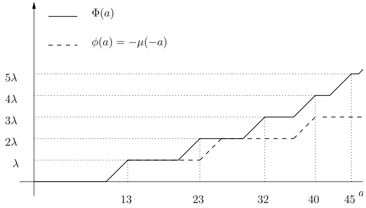

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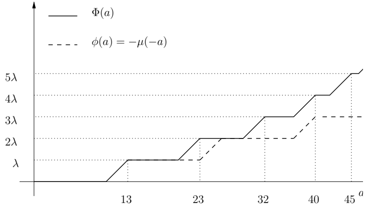

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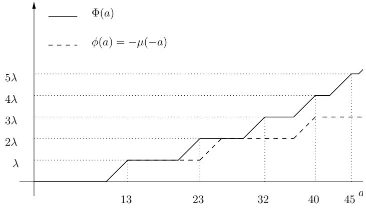

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1. Superadditive lifted gives strong ineqs, but it is cumbersome to compute lifting functions
2. Two-step MIR gives exact superadditive lifted inequalities if $u = \infty$
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