Planning Activities with Start-Time Dependent Variable Costs

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Multiple Period Planning Model

Time Horizon: \( T \) periods
Activities to Plan: \( a \in A \)
Decision Variables: \( x_a = (x_{a1}, \ldots, x_{aT}), \ a \in A \)
\( u \in \mathbb{R}^K \)
Feasible Region: \( x_a \in X_a \subseteq [0, M_a]^T \)
\( (x, u) \in C \subseteq \mathbb{R}^{T|A|+K} \)

The Model

\[
\begin{align*}
\min & \quad \sum_{a \in A} h_a(x_a) + \sum_k d_k u_k \\
\text{s.t.} & \quad x_a \in X_a \quad \forall a \in A \\
& \quad (u, x) \in C.
\end{align*}
\]
The Cost Function for Each Activity is Non-convex

\[ c_{at} : \text{ Variable cost of activity } a \text{ over the entire horizon if the activity begins in period } t \]

Assumption (improving technology): \( c_{a1} \geq c_{a2} \geq \cdots \geq c_{aT} \)

**The Cost Function**

\[ h_a(x) = \sum_{s=1}^{T} 1 (\min \{ k : x_k > 0 \} = s) c_{as} \sum_{t=s}^{T} x_t \]

Note: \( h_a \) is concave over \( \mathbb{R}_+^T \) and discontinuous.

Our Approach: Develop strong formulations for single activity; that is, for fixed \( a \), consider formulation with cost function \( h_a(x) \) and \( x \in X_a \).
Example Cost Function $h(x)$ for $T = 2$

$$c_1 = 1, \ c_2 = \frac{1}{2}, \ h(x_1, x_2) = 1(x_1 > 0)(x_1 + x_2) + 1(x_1 = 0)\frac{1}{2}x_2$$
Compact Formulation and Specialized Branching

- Introduce no auxiliary modeling variables.
- Linear lower bound on objective function:
  \[ h(x) \geq \sum_{t=1}^{T} c_t x_t \]

- Branching on start-time \( s \):
  - Left branch: \( s \leq k \). Update objective lower bound.
  - Right branch: \( s > k \). Fix \( x_t = 0, t = 1, \ldots, k \).

- Requires implementation of branching.
- Cuts can be used to get stronger objective lower bound.
Introduce auxiliary variables

\( z_t \): Amount of activity that is charged at rate \( c_t \); i.e. amount “produced” in period \( t \) that can be used in periods \( s \geq t \)

All activity must be charged at rate in period activity starts

\( \Rightarrow z_t \) positive in at most one period (SOS1)

Replace \( h(x) \) with \( \sum_{t=1}^{T} c_t z_t \) in objective, and add constraints:

\[
\sum_{s=1}^{t} z_s \geq \sum_{s=1}^{t} x_s \quad t = 1, \ldots, T
\]

\( \{z_t : 1 \leq t \leq T \} \) SOS1  \( \quad (5) \)

Note: (5) can also be enforced by adding binary variables.
Extended Formulation Inspired by Lot Sizing

Introduce auxiliary variables

\( w_{st} \): Amount of activity that is charged at rate \( c_s \)
and performed in period \( t \geq s \)

Replace \( h(x) \) with \( \sum_{s=1}^{T} c_s \sum_{t=s}^{T} w_{st} \) in objective, and add:

\[
\sum_{s=1}^{t} w_{st} = x_t \quad t = 1, \ldots, T \quad (6)
\]

\[
w_{st} \leq Mb_s \quad s = 1, \ldots, T \quad (7)
\]

\[
\sum_{t=1}^{T} b_t \leq 1 \quad (8)
\]

\[
b_t \in \{0, 1\} \quad t = 1, \ldots, T \quad (9)
\]

Note: For a single activity, this formulation is integral.
Strengthening the Formulations: A Special Case

- \( X = \{ x : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_T \leq M \} \).
- The nondecreasing constraint was present in the motivating application.
- Compact formulation: Improved lower bound

\[
  h(x) \geq \sum_{t=1}^{T} c_t x_t + \sum_{t=1}^{T-1} (T - t)(c_t - c_{t+1})x_t \tag{10}
\]

- Lot sizing inspired formulation: Strengthen inequalities (4)

\[
  \sum_{s=1}^{t} z_s \geq \sum_{s=1}^{t} x_s + (T - t)x_t \quad t = 1, \ldots, T \tag{11}
\]
Theorem

\[ \text{conv}(F) = E = \text{Proj}_{(\mu, x)}(P) \]

where \( F = \{ (\mu, x) \in \mathbb{R} \times X : \mu \geq h(x) \} \) is the non-convex feasible region (the epigraph of non-convex function \( h \)),

\[ E = \{ (\mu, x) \in \mathbb{R} \times X : \mu \geq \sum_{t=1}^{T} c_t x_t + \sum_{t=1}^{T-1} (T - t)(c_t - c_{t+1}) x_t \} \]

is the strengthened compact formulation, and

\[ P = \{ (\mu, x, z) \in \mathbb{R} \times X \times \mathbb{R}^T_+ : \mu = \sum_{t=1}^{T} c_t z_t, (x, z) \text{ satisfy (11)} \} \]

is the strengthened lot sizing inspired formulation.
Select Computational Results: Production and Distribution Planning

- Minimize costs to meet demand over the planning horizon. Production and distribution costs are start-time dependent.
- Instances randomly generated, with characteristics similar to real data.

Results for lot sizing formulation with binary variables: strengthening the formulation with (11) is crucial.

| $|A|$ | $T$ | Ineqs (4) | Ineqs (11) | Time(s) or * Gap | Nodes |
|---|---|---|---|---|---|---|
| 100 | 10 | 50.3 | 8.2 | 451 | 5 |
| 150 | 10 | * 0.08% | 13.4 | 75488 | 76 |
| 200 | 10 | * 0.09% | 45.2 | 35951 | 242 |
| 75 | 15 | 992.6 | 9.9 | 29738 | 39 |

* Did not finish after limit of 1 hour.
Good Solutions Can be Found for Large Instances

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<th>A</th>
<th>T</th>
<th>No-A</th>
<th>LS-S</th>
<th>LS-B</th>
<th>No-A</th>
<th>LS-S</th>
<th>LS-B</th>
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<td>* 0.38%</td>
<td>* 0.18%</td>
<td>6891</td>
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* Did not finish after limit of 1 hour.

No-A = Compact formulation, LS-S = Lot sizing with SOS1, LS-B = Lot sizing with binaries
Compact formulation has significantly faster LP solve times.
The Approach Can Also Handle Side Constraints

- Add semi-continuous restrictions on activities:
  \( x_t \in \{0\} \cup [l, M] \)
- For compact and lot sizing with SOS1, binaries added only to model this restriction

<table>
<thead>
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<th>( A )</th>
<th>( T )</th>
<th>Time(s) or * Gap</th>
<th>Nodes</th>
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<td>* 0.97%</td>
<td>* 1.01%</td>
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<td>15</td>
<td>* 0.49%</td>
<td>* 0.56%</td>
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<tr>
<td>400</td>
<td>15</td>
<td>* 0.20%</td>
<td>* 0.29%</td>
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For larger instances, formulation with binaries yields better gap within time limit.
Extensions and Ongoing Research

- Remove the non-decreasing constraint on activities. That is, let

\[ X = \left\{ x \in \mathbb{R}_+^T : x_t \leq M, \ t = 1, \ldots, T \right\}. \]

Single activity convex hull defined by exponential family of inequalities in all formulations (except extended).

- Incorporate fixed cost for installing technology. Motivates further study of models with binary variables present.

- Consider more complicated sets \( X \). For example, time-dependent upper bounds, or production ramping constraints.