Mixed-Integer Nonconvex problems: an MILP perspective

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Lehigh University – September 11, 2008

Mixed-Integer Non-Linear Programming

$$
\begin{array}{llll}
\mathbf{P}_0) & \min & f(x) & \\
& s.t. & g_i(x) \leq 0 & i \in I \\
& & x_j^l \leq x_j \leq x_j^u & j \in N_0 = 1, 2 \ldots, n \\
& & x_j \in \mathbb{Z} & j \in J_0 \subseteq N_0
\end{array}
$$

- f and g_i's are, in general, nonconvex
- If f and g_i's are convex, we call $\mathbf{P}_0$ a **convex** MINLP
- f and g_i's are factorable: can be written as $\sum_{i=1}^{k} \prod_{j=1}^{p} h_{ij}(x)$, with h_{ij} univariate with factorable arguments

Applications

- Water treatment: Design of water networks with reuse of water, decentralized water treatment (minimize the consumption of fresh water)
- Scheduling and blending for production plants: coupling the problem of scheduling the production in a refinery and blending operations to get gasoline of different grades
- Trimloss problems for paper, wood, film, steel, glass industry – Mixed Integer bilinear problems (two sets of variables, formulation is linear in each set individually)
- Portfolio optimization. Convex in the classical case, but discrete if there are transaction (fixed) costs and nonconvex if robustness is introduced

Previous work

- Branch & Bound (B&B) (Gupta & Ravindran ’85; Tuy & Horst ’88; Nabar & Schrage ’91; Borchers & Mitchell ’94; Stubbs & Mehrotra ’99)
- Generalized Benders Decomposition (Geoffrion ’72)
- Outer-Approximation (Duran & Grossmann ’86; Yuan et al. ’88; Fletcher & Leyffer ’94)
- LP/NLP based B&B (Quesada & Grossmann ’92)

Software:

- Baron (Tawarmalani & Sahinidis)
- LaGO (Nowak & Vigerske)
- (convex) Bonmin (Bonami et al.), FilMINT (Abhishek, Leyffer, Linderoth)

How do we solve it?

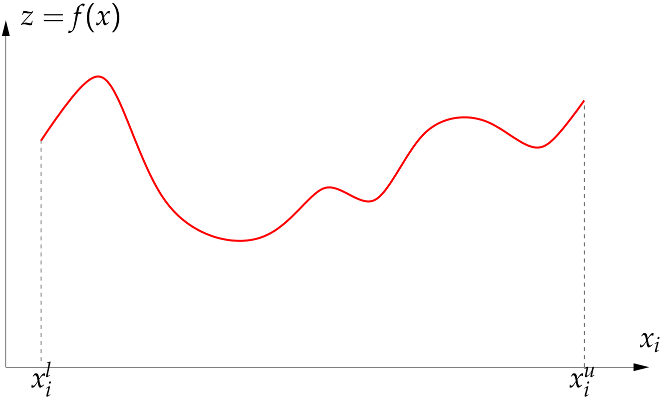
With a spatial Branch&Bound[1]: enumerate implicitly all local minima, use a convex (linear) relaxation to find lower bounds.

Key components:

- linearization (or convexification) for lower bounds
- heuristics for upper bounds
- branching rules to partition the solution set
- bound tightening to reduce the solution set

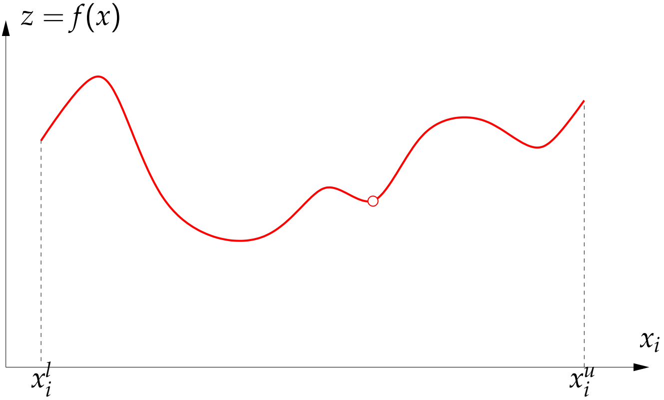
[1] See also Smith&Pantelides 1997, Tawarmalani&Sahinidis 2002

Solving nonconvex MINLPs

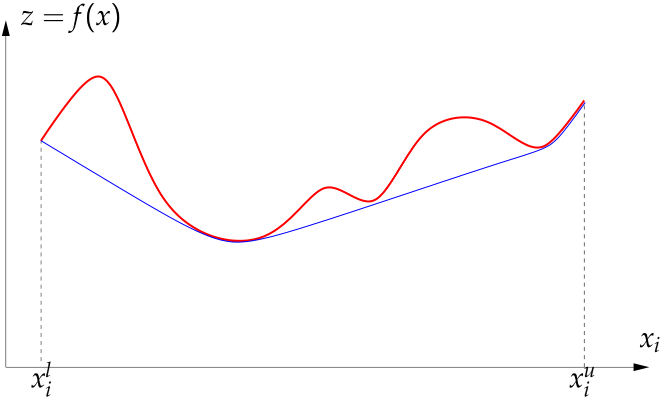


Relaxing integrality → nonconvex NLPs

Solving nonconvex MINLPs

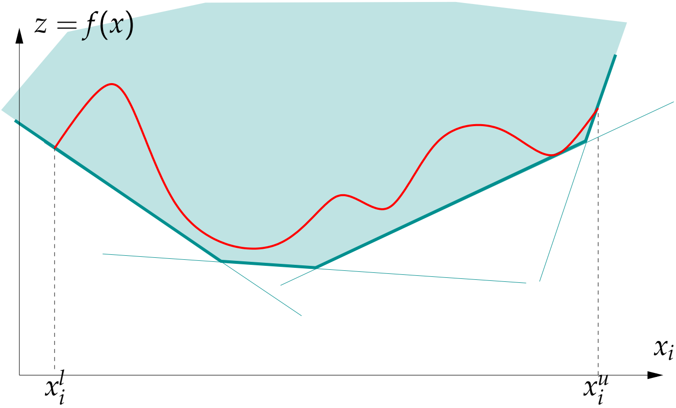


$\Rightarrow$ finding a valid lower bound is difficult (local minimum)



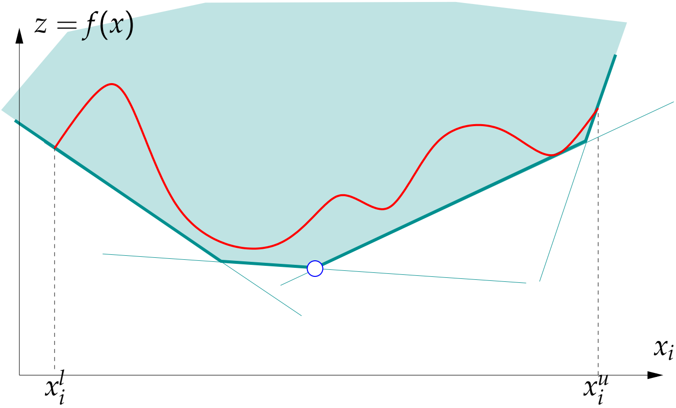
Usually, a convex relaxation is sought

Solving nonconvex MINLPs



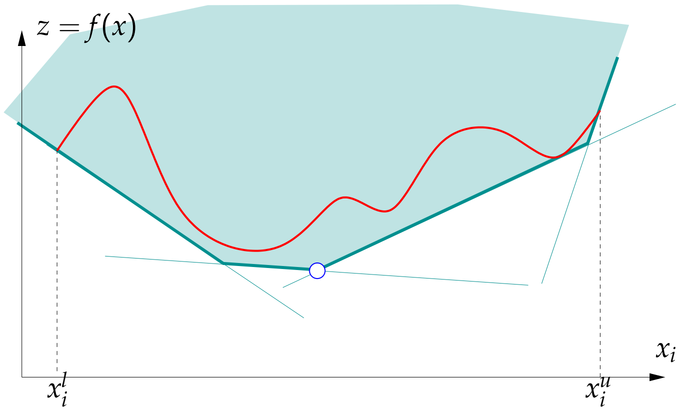
For instance, a **linear** relaxation

Solving nonconvex MINLPs



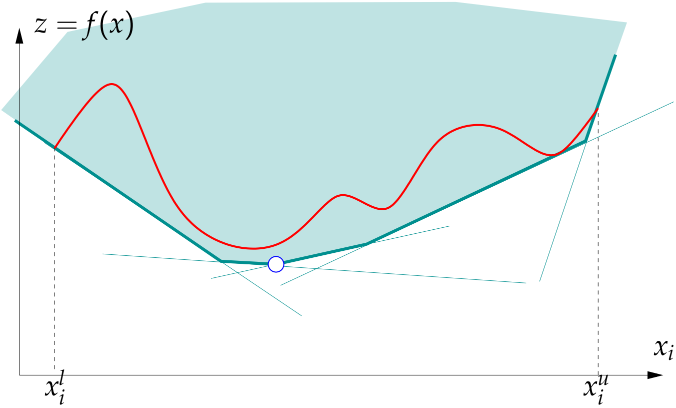
$\Rightarrow$ get a lower bound

Solving nonconvex MINLPs



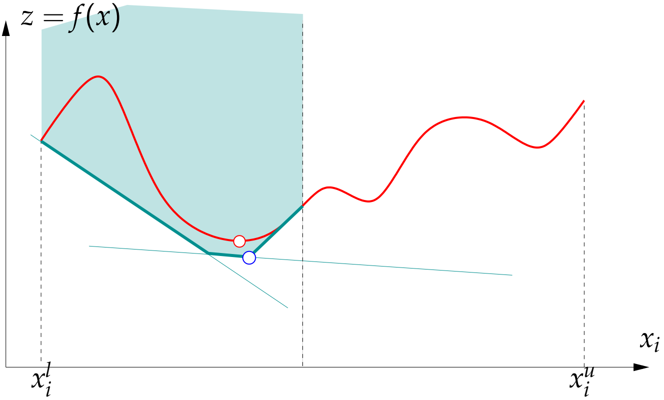
Solution may be NLP-infeasible (and/or fractional)

Solving nonconvex MINLPs



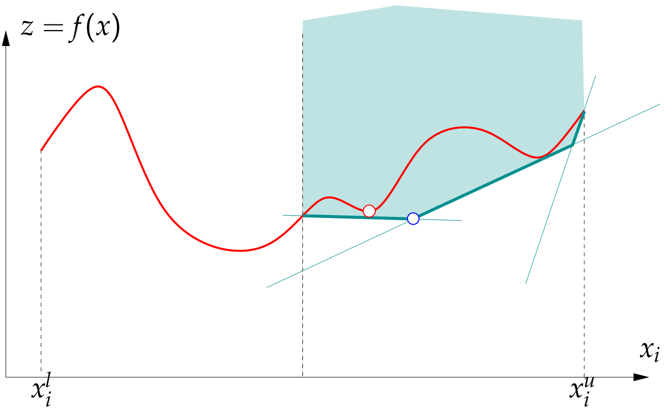
Either refine the linearization

Solving nonconvex MINLPs



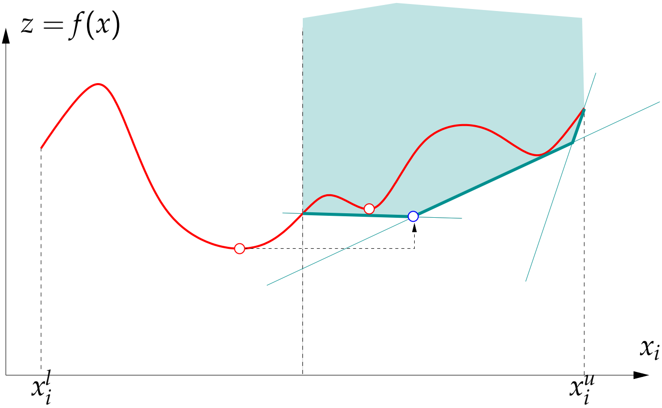
or branch on continuous variables

Solving nonconvex MINLPs



Linearization and lower bound improves

Solving nonconvex MINLPs



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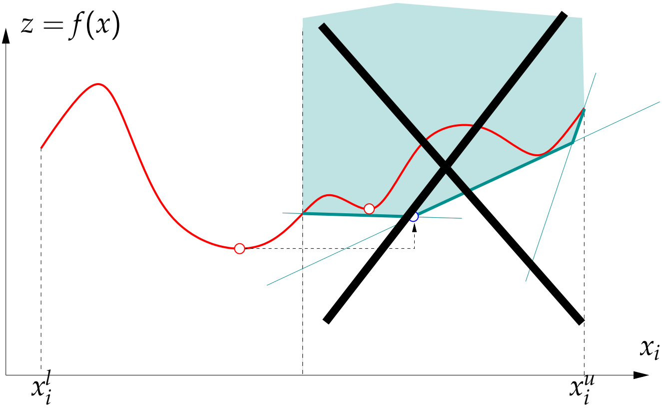

Linearization and lower bound improves

Couenne, a solver for nonconvex MINLPs

Couenne[2] is a Branch&Bound for nonconvex MINLPs. Written in C++, available as Open Source in Coin-OR (www.coin-or.org), it implements

- linearization of nonconvex functions
- heuristics for upper bound
- specialized branching rules
- bound tightening

It uses code from `Bonmin` (MINLP B&B), `Cbc` (Branch&Bound), `Cgl` (Cut generation), `Clp` (LP solver), `Ipopt` (NLP solver), and `LaGO` (quadratic forms).

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[2] **C**onvex **O**ver/**U**nder **EN**velopes for **N**onlinear **E**stimation

A late outline

Already there

- linearization
- branching rules

Soon there

- disjunctive cuts
- nonconvex feasibility pump
- linearization cuts for MIQQP problems

Convex relaxations of non-convex MINLPs

$\mathbf{P}_0$ factorable $\Rightarrow$ can be reformulated (Smith&Pantelides, 1997)

- $\sum_{i=1}^k h_i(x)$ becomes $\boxed{\begin{array}{l}\sum_{i=1}^k x_{n+i} \\ x_{n+i} = h_i(x), 1! \leq i \leq k\end{array}}$
- $\prod_{i=1}^k h_i(x)$ becomes $\boxed{\begin{array}{l}\prod_{i=1}^k x_{n+i} \\ x_{n+i} = h_i(x), 1 \leq i \leq k\end{array}}$
- $h_1(h_2(x))$ becomes $h_1(x_2)$, with $x_2 = h_2(x)$
- $\ldots$

Recursively apply until all nonlinear constraints are of the form $x_k = \vartheta_k(x_1, x_2 \ldots, x_{k-1})$, with $\vartheta_k \in \Theta = \{\sum, \prod, \exp, \log, \sin, \text{abs} \ldots\}$.

Reformulation

The initial problem

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& & x_j \in \mathbb{Z} & j \in J_0 \subseteq N_0
\end{array}
$$

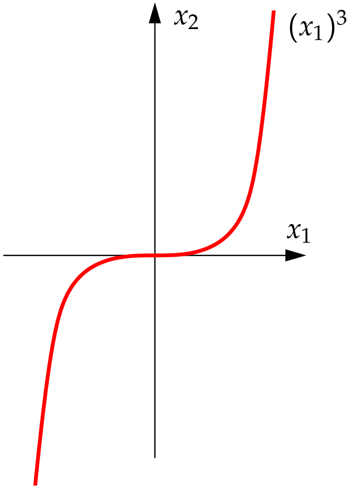
is **reformulated** as an equivalent problem

$$
\begin{array}{llll}
\mathbf{P}) & \min & x_{n+q} & \\
& s.t. & x_k = \vartheta_k(x_1, x_2 \ldots, x_{k-1}) & k = n+1, n+2 \ldots, n+q \\
& & x_j^l \leq x_j \leq x_j^u & j \in N = 1, 2 \ldots, n+q \\
& & x_j \in \mathbb{Z} & j \in J \subseteq N
\end{array}
$$

Then, each $x_k = \vartheta_k(x_1, x_2, \ldots, x_{k-1})$, $k = n+1, n+2 \ldots, n+q$, is linearized through inequalities $a_k x_k + A_k x \geq b_k$.

Linearization

$x_2 = \vartheta(x_1) = (x_1)^3$
with $x_1^l \leq x_1 \leq x_1^u$



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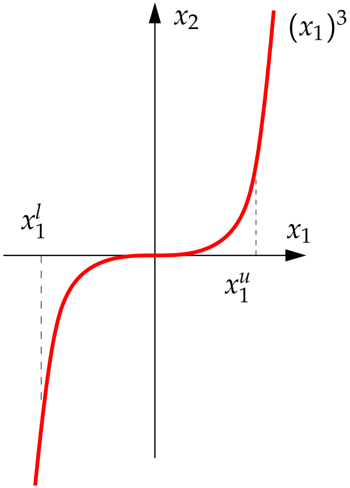

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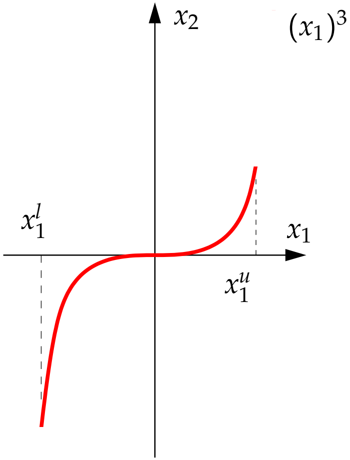

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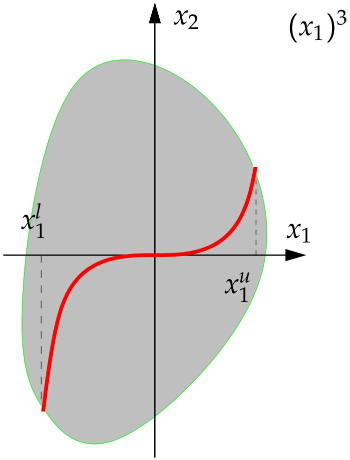

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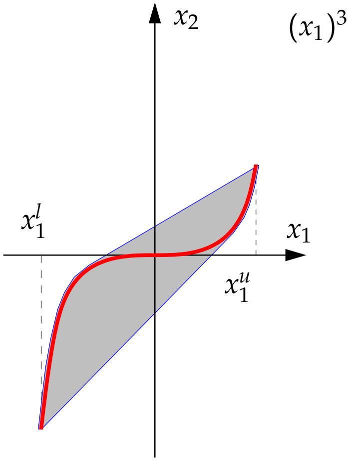

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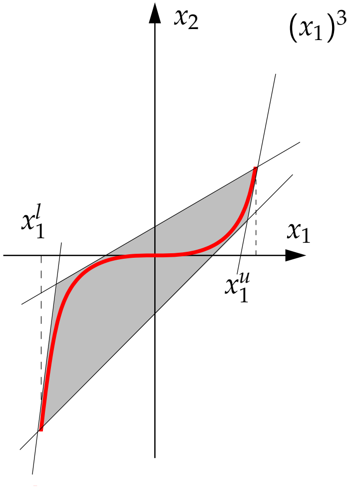

Linearization

Obtain the equivalent problem

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& s.t. & x_k = \vartheta_k(x_1, x_2 \ldots, x_{k-1}) & k = n+1, n+2 \ldots, n+q \\
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& & x_j \in \mathbb{Z} & j \in J \subseteq N
\end{array}
$$

Replace each $x_k = \vartheta_k(x_1, x_2, \ldots, x_{k-1})$, $k = n+1, n+2, \ldots, n+q$ with inequalities $a_k x_k + A_k x \geq b_k$.

$$
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\mathbf{LP}) & \min & x_{n+q} & \\
& s.t. & a_k x_k + A_k x \geq b_k & k = n+1, n+2 \ldots, n+q \\
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\end{array}
$$

A linear relaxation providing a valid lower bound.

Spatial B&B with convexification

At each node:

(Lb) **repeat** k times:

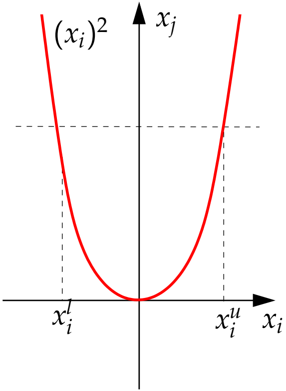
(⋆) add linearization cuts

if no cuts found, **break**

get *lower bd.*, soln. $\hat{x}$

(Ub) Look for a *feasible solution* with NLP solver

(Br.) If $x_k = \theta_k(x)$ *infeasible*, i.e. $\hat{x}_k \neq \theta_k(\hat{x})$, branch on x



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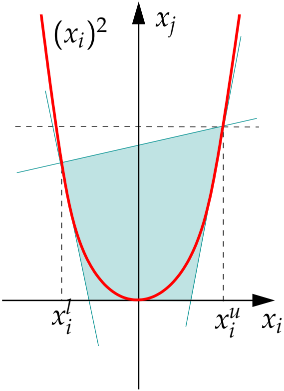

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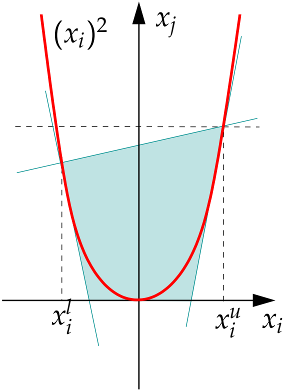

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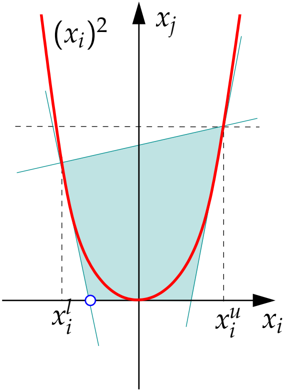

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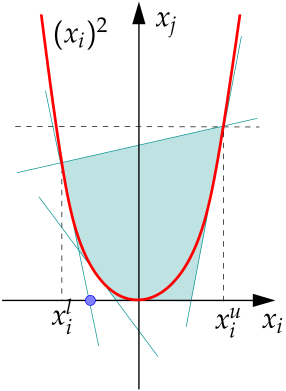

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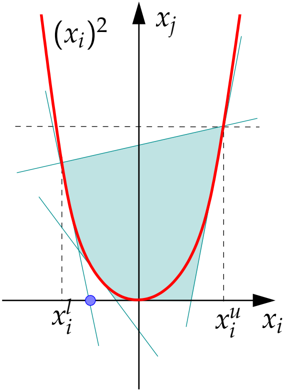

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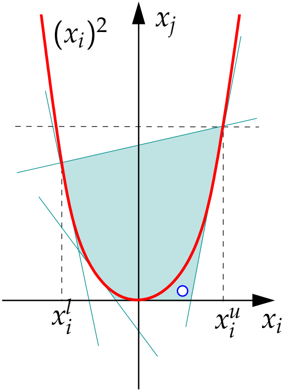

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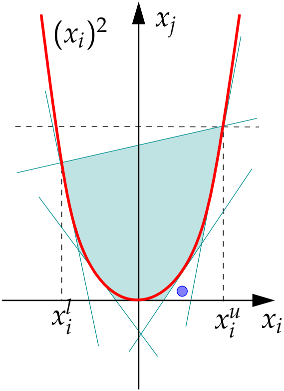

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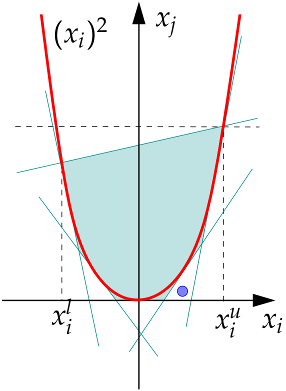

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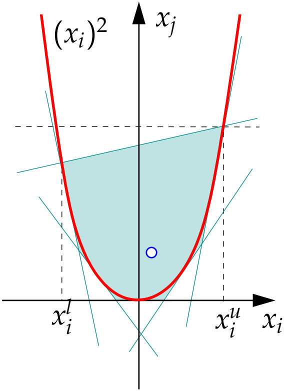

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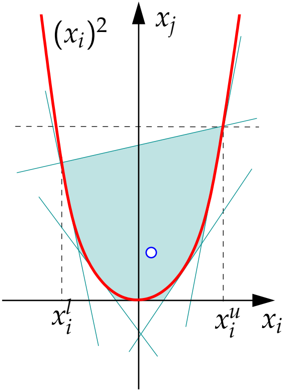

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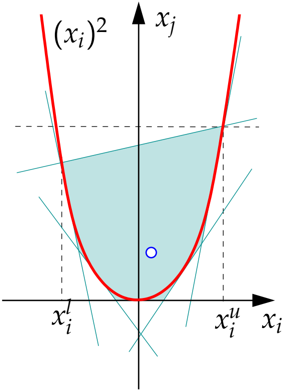

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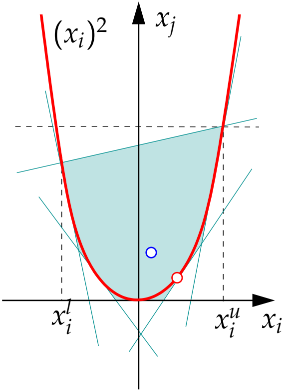

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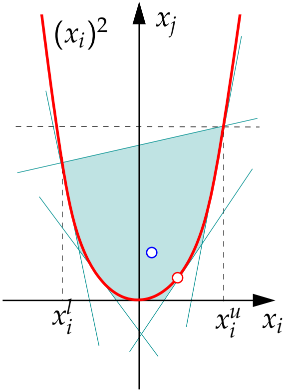

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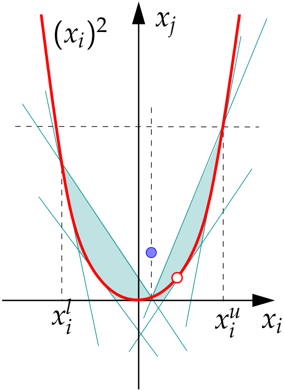

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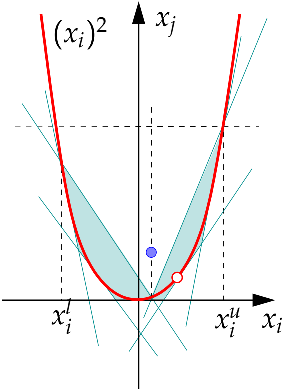

($\star$) solves a **separation** problem: separate current iterate from **convex envelope of** $\{x \in \mathbb{R}^{n+q} : x_k = \theta_k(x)\}$

Branching rules

Purpose: partition the solution set in $\geq$ two (easier) subproblems.

- MILP: if a component $x_i^\star$ of the LP solution is fractional, create subproblems: P_1 with branching rule $x_i \leq \lfloor x_i^\star \rfloor$; P_2 with $x_i \geq \lceil x_i^\star \rceil$
- MINLP: may be necessary for continuous variables.

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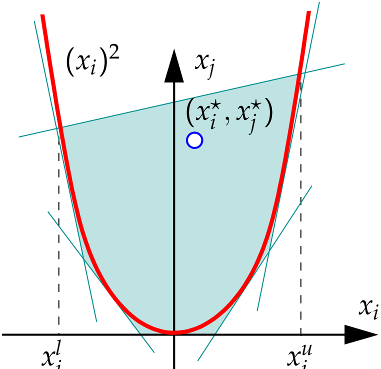

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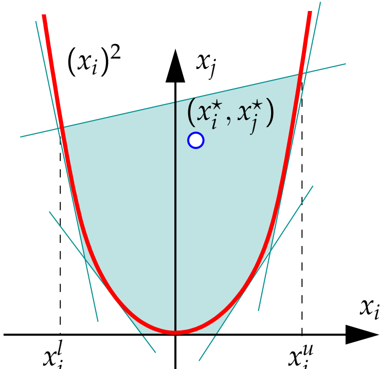


Strong branching

Strong branching[3]: for each branching candidate x_i,

- simulate br. rule $x_i \leq x_i^b$, re-solve $\to$ new lower bound $x_{n+q}^{\leq}$
- simulate br. rule $x_i \geq x_i^b$, re-solve $\to$ new lower bound $x_{n+q}^{\geq}$
- set $U_i^{\text{strong}} = \alpha \min\{x_{n+q}^{\leq}, x_{n+q}^{\geq}\} + (1-\alpha)\max\{x_{n+q}^{\leq}, x_{n+q}^{\geq}\}$, with $0 < \alpha < 1$

Choose variable x_i with maximum U_i^{strong}

Computationally expensive...
$\Rightarrow$Pseudocosts[4], Reliability Branching[5]: statistics on U_i^{strong} at initial nodes are used to *estimate* it at later nodes

[3] Applegate et al., "The TSP, a computational study".
[4] Benichou et al, "Experiments in MIP", *MathProg* '71
[5] Achterberg et al., "Branching rules revisited", *OR letters* 2005

Comparing with Baron

Baron (Branch And Reduce Optimization Navigator) is currently the state-of-the-art MINLP solver (Tawarmalani&Sahinidis 2002).

- A spatial branch&bound with linearization, bound reduction, and heuristics
- Uses external LP and NLP solvers (Cplex 9 and Minos)
- Couenne with reliability branching

Comparing with Baron – MINLP problems

Name	#var	#int	#con	Couenne time (lb)	Couenne ub	Baron time (lb)	Baron ub
non-convex MINLP							
Multistage	185	39	265	(-17621.4)	–	70.88	-7581
barton-aiche1	818	66	987	(-102.47)	–	(-103.31)	-81.8659
c-sched-4-7	233	168	138	(-254146)	–	(-1.93e+05)	-1.33e+05
ex1233	48	12	52	(76225.9)	161022	169.80	1.55e+05
ex1243	57	29	75	4.95	83402.5	1.33	83402.5
ex1244	86	40	110	(68674.3)	85431.1	25.01	82040.0
ex1252	39	15	43	124.50	128894	0.23	128894
nous1	48	2	41	(1.510)	1.567	169.20	1.567
nous2	48	2	41	277.26	0.626	1.22	0.626
nConvPl	948	148	920	(-8790.4)	-4580	(-8946.17)	-7529.3
space-25	893	750	235	(84.61)	–	(155.566)	784.84
space-25-r	818	750	160	(71.72)	–	(160.507)	786.34
feedloc	89	96	247	114.54	0	2.93	0
MIQQP[6]							
ibell3a	122	209	104	1164.90	878785	(-3.36e+09)	2966916.97
ibienst1	505	83	576	4065.70	48.74	(-2.42e+09)	48.74
imisc07	260	957	212	(2501.63)	2814.28	–	–
iran8x32	512	767	296	4643.10	5255.45	–	–
conic (convex) MINLP[7]							
classical_40_0	120	41	83	1233.30	-0.0815	218.56	-0.0815
classical_40_1	120	41	83	98.04	-0.0847	20.75	-0.0847
robust_20_0	83	22	66	4.42	-0.0798	2.60	-0.0798
robust_20_1	83	22	66	25.85	-0.0533	16.39	-0.0533
shortfall_20_0	84	22	67	56.89	-1.090	6.90	-1.090
shortfall_20_1	84	22	67	(-1.076)	-1.066	35.79	-1.075

[6] H. Mittelmann, `http://plato.asu.edu/ftp/miqp`

[7] Vielma, Ahmed, & Nemhauser 2008

Comparing with Baron – pure NLP problems

			Couenne		Baron	
Name	#var	#con	time (lb)	ub	time (lb)	ub
Hicks_5	83	68	7.37	227.26	21.89	227.26
Hicks_20	338	278	110.45	227.26	334.40	229.7
Hicks_50	848	698	755.10	227.26	3968	227.26
Hicks_100	1698	1398	3994.40	227.26	–	–
ex5_2_5	33	19	(-7211.96)	-3500	(-5055)	-3500
ex5_3_3	62	53	(1.89745)	3.234	(2.174)	3.234
foulds3	168	48	(-59.7432)	-8	(-69.809)	-8
QCQP						
dualc8	9	16	(18309.0)	18309.2	(18306.3)	18309.1
dual1	86	2	(-205.22)	0.035	(-176.46)	0.035
dual4	76	2	–	–	(-198.742)	0.746
qadlittl	97	54	3775.80	480319	216.93	480319
qp1	50	2	(-0.0831)	8.093e-4	(-0.304)	8.093e-4
qp2	50	2	(-0.0891)	8.093e-4	(-0.305)	8.093e-4
qp3	100	52	(-0.2905)	8.093e-4	(-0.093)	8.093e-4
cvxqp1_s	101	51	(10767.4)	12467.9	(9739.53)	11590.7
cvxqp2_s	101	26	(7298.61)	–	(6828.31)	8120.94
cvxqp3_s	101	76	(11943)	–	166.25	11943.4
primal4	1490	76	1650.50	-0.746	(-0.779)	0
qetamacr	543	334	2194.80	86760.4	(61835.2)	86760.4
gouldqp2	700	350	(-0.165)	1.84e-4	(-0.186)	1.84e-4
qisrael	143	164	686.72	2.5347e+7	78.92	2.53e+07
qshare1b	221	111	(720058)	–	983.35	720078
stcqp1	3159	1	1598.70	155144	(148327.34)	157758.85
values	203	2	840.27	-1.39	(-12.90)	-1.39

Comparing with Baron – box QP problems

		Couenne		Baron	
Name	#var	time (lb)	ub	time (lb)	ub
spar030-060-1	30	2081.10	-706	(-830.759)	-706
spar030-060-2	30	24.43	-1377.17	3.62	-1377.17
spar040-050-1	40	(-1188.02)	-1154.5	(-1403.72)	-1154.5
spar040-050-2	40	4053.90	-1430.98	(-1636.44)	-1430.98
spar040-060-1	60	(-1718.03)	-1311.06	(-2009.22)	-1322.67
spar050-040-1	60	(-1597.44)	-1411	(-1900.97)	-1411
spar050-050-1	70	(-2203.13)	-1193	(-2685.19)	-1198.41
spar060-020-1	80	470.49	-1212	3582.38	-1212
spar060-020-2	90	45.11	-1925.5	55.06	-1925.5
spar070-025-1	70	(-2856.69)	-2538.91	(-3169.19)	-2538.91
spar080-025-1	80	(-3758.34)	-3157	(-4173.78)	-3157
spar090-025-1	90	(-4861.59)	-3361.5	(-5468.25)	-3372.5

Improving Couenne

As Open Source code, **Couenne** can be used as a *base* for the development and test of MINLP models/solvers, by

- specializing linearization, branching, heuristic, and bound reduction techniques
- extending the set of operators: quadratic forms, polynomials…
- generalizing MILP techniques to MINLP

Augmenting the set of operators

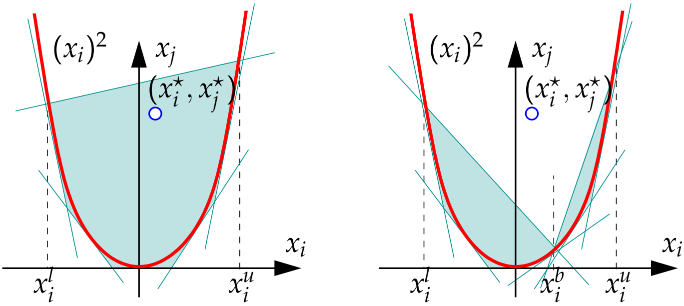
- any function $f : \mathbb{R}^n \to \mathbb{R}$ can do
- must provide procedure to **separate** a point $x \in \mathbb{R}^{n+1}$ from

$$\mathrm{convenv}\{x \in \mathbb{R}^{n+1} : x_{n+1} = f(x_1, x_2 \ldots, x_n)\}$$

```
class polydeg4:  public expression {

    [...]

    double compute(double x) {
        return myComputePoly4 (x);
    }

    void generateCuts(double *curpoint) {
        // separation procedure
    }
}
```

MILP extension #1: Disjunctive cuts

- Disjunctions arise naturally in Integer Programming and also in nonconvex MINLP!
- Recently used in non-MILP contexts – e.g. Saxena et al., IPCO 2008, for MIQQPs: disjunction from $X - xx^T \preceq 0$
- In nonconvex (MI)NLP, disjunction are provided by branching rules on nonlinear expressions.



Disjunctive cuts for nonconvex MINLP

Consider a nonconvex MINLP and its current linearization, $\min\{x_{n+q} : Ax \leq a, l \leq x \leq u\}$. Optimal LP solution is $x^\star$.

- branching rule on $x_i \Rightarrow$ two *refined* linearizations:
  - $x_i \leq b$: extra inequalities $Bx \leq b$
  - $x_i \geq b$: extra inequalities $Dx \leq d$
- create the *Cut Generating LP*, find deepest cut $\alpha x \leq \beta$:

$$
\begin{array}{llllllll}
\max & \alpha x^\star & -\beta & & & & & \\
 & \alpha & & = uA & & +u'B & & \\
 & \alpha & & = & vA & & +v'D \\
 & & \beta & \geq ua & & +u'b & & \\
 & & \beta & \geq & va & & +v'd \\
 & & & \multicolumn{5}{l}{||(u, v, u', v')||_1 = 1}
\end{array}
$$

- cons: one huge LP for just one cut?

MILP extension #2: Nonconvex Feasibility Pump

- principle: two sequences of points
  - $\hat{x}^k$ Integer but infeasible for the relaxation
  - $\bar{x}^k$ Fractional but feasible for the relaxation
- Originally introduced for MILP (Fischetti, Glover, Lodi)
- Extended to convex MINLP (Bonami, Cornuéjols, Lodi, Margot)

MILP extension #2: Nonconvex Feasibility Pump

FP for MILP (Fischetti et al.):
$\min\{c^T x : Ax \geq b, x_i \in \mathbb{Z} \forall i \in J \subseteq N\}$

- $\bar{x}^0 = \mathrm{argmin}\{c^T x : Ax \geq b\}$; let $k \leftarrow 0$
- **while** $\bar{x}^k$ not integer

  let $\hat{x}^k := \lfloor \bar{x}^k \rceil$

  let $\bar{x}^{k+1} := \mathrm{argmin}\{||x - \hat{x}^k||_1 : Ax \geq b\}$

  let $k \leftarrow k+1$

MILP extension #2: Nonconvex Feasibility Pump

FP for convex MINLP (Bonami et al.):
$\min\{f(x) : g(x) \leq 0, x_i \in \mathbb{Z} \forall i \in J \subseteq N\}$

- $\bar{x}^0 = \text{argmin}\{f(x) : g(x) \leq 0\}$; let $k \leftarrow 0$
- **while** $\bar{x}^k$ not integer

  let $\hat{x}^k := \text{argmin}\{||x - \bar{x}^k||_1 : Ax \geq b, x_i \in \mathbb{Z} \forall i \in J \subseteq N\}$

  let $\bar{x}^{k+1} := \text{argmin}\{||x - \hat{x}^k||_1 : g(x) \leq 0\}$

  let $k \leftarrow k + 1$

[$Ax \geq b$ is an **Outer Approximation** of the problem]

FP extends naturally to nonconvex MINLP, if Outer Approximation is replaced by linearization inequalities.

Current work #3: linearization of MIQQP

Joint work with F. Margot, A. Qualizza (CMU)

- consider the MIQQP: $\min\{x^T Q_0 x + a_0^T x : x^T Q_i x + a_i^T x \leq b_i\}$, Q_i not PSD (in general)
- reformulate: $\min\{Q_0 \bullet X + a_0^T x : Q_i \bullet X + a_i^T x \leq b_i, X = xx^T\}$
- $X = xx^T$ means $X - xx^T \succeq 0$ and $X - xx^T \preceq 0$
- $X - xx^T \succeq 0$ equals $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \tilde{X} \succeq 0$
- Linearize $\tilde{X} \succeq 0$ by separating[8] cuts of the form $a^T \tilde{X} a \geq 0$ for any vector a

[8] See also Sherali & Fraticelli, Sivaramakhrishnan & Mitchell…

Future work

Interface to generate problems at code level (analogous to Ilog's Concert Technology or GLPK C++ interface)

```
int main (int argc, char **argv) {

    CouenneVar x1, x2;

    CouenneConstraint c1 = x1^2 + x2^2 <= 1;
    CouenneMinObj o1 = 2*x1 + 3*x2;

    CouenneProblem p;
    p << x1 << x2 << o1 << c1;
    p.solve();

    CouenneConstraint c2 = x2 >= .5;
    p << c2;
    p.resolve();
}
```

Resources

- P. Belotti, J. Lee, L. Liberti, F. Margot, A. Wächter, "Branching and bounds tightening techniques for non-convex MINLP," `opt-online` (Sep. 2008).

http://egon.cheme.cmu.edu/ibm/page.htm

https://projects.coin-or.org/Bonmin/wiki/BonCouenne

(or google "boncouenne")