Branching on general disjunctions:

Complexity and Computational Experiments

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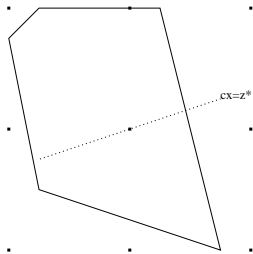Department of Industrial and Systems Engineering
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Informs 2008

COR@L

COMPUTATIONAL OPTIMIZATION
RESEARCH AT LEHIGH

Branch and Bound Procedure

MIP: $\min_{x \in S} cx$

$S: \begin{array}{ccc} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$

LP: $\min_{x \in \mathcal{P}} cx$

$\mathcal{P}: \begin{array}{ccc} Ax & \geq & b \\ x & \in & \mathbb{R}^n, \end{array}$
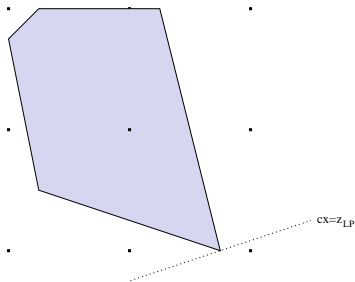
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z_{LP}

$cx=z_{LP}$

Branch and Bound Procedure

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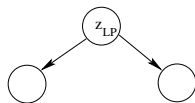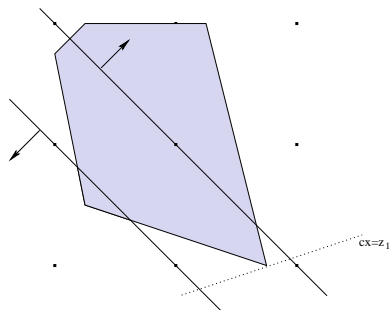
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$cx=z_2$

$cx=z_3$

Branch and Bound Procedure

MIP: $\min_{x \in S} cx$

$$S : \begin{array}{ccc} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

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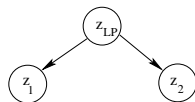$$\mathcal{P} : \begin{array}{ccc} Ax & \geq & b \\ x & \in & \mathbb{R}^n, \end{array}$$

Branch and Bound Procedure

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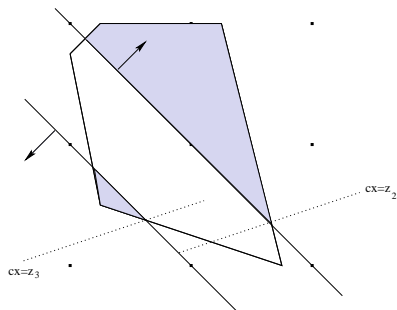
Branch and Bound Procedure

MIP: $\min\limits_{x \in S} cx$

$S: \begin{array}{rcl} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$

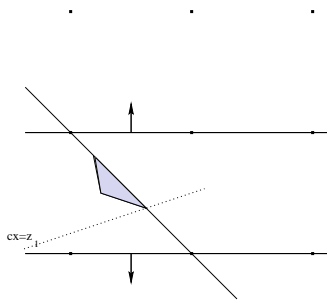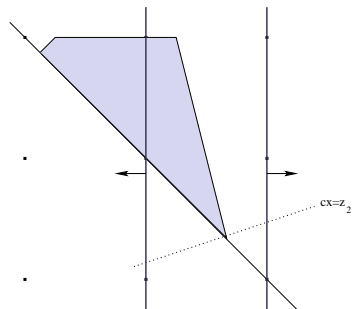LP: $\min\limits_{x \in \mathcal{P}} cx$

$\mathcal{P}: \begin{array}{rcl} Ax & \geq & b \\ x & \in & \mathbb{R}^n, \end{array}$

Branching Decision

- ▶ Any subsets S_1, S_2 such that $S \subseteq (\mathcal{P} \cup S_1) \cup (\mathcal{P} \cap S_2)$ could be used to partition
- ▶ We restrict ourselves to partitions that are polyhedra
- ▶ Assume that we only create at most two partitions in each iteration
- ▶ The branching rule $x_i \leq \pi_0 \vee x_i \geq \pi_0 + 1$ $i = 1, \ldots, d$ is valid. (Variable disjunction)
- ▶ A more general valid disjunction is $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$, where $(\pi, \pi_0) \in \mathbb{Z}^d \times 0^{n-d} \times \mathbb{Z}$. (General disjunction)
- ▶ e.g. $2x_1 + x_2 \leq 4 \vee 2x_1 + x_2 \geq 5$

Previous Work

Branching on Variables:

- ▶ Finding the minimum size tree is $\mathcal{NP}$−Hard. (Liberatore, 2000)
- ▶ Heuristics for improving bounds: Strong branching, Pseudo-cost branching, Reliability branching (Experiments by Linderoth and Savelsbergh, 1999, Achterberg et al., 2005)
- ▶ Heuristics for feasible solutions (Patel and Chinneck, 2007)
- ▶ Heuristics for conflict analysis and resolution (Chvátal, 1997, Achterberg, 2007)
- ▶ Heuristics based on thin directions (Derpich and Vera, 2006)
- ▶ Information theoretic approaches (Gilpin and Sandholm, 2007)

Branching on General Disjunctions:

- ▶ Trees polynomial in size, when dimension is fixed (Lenstra 1983, other work by Aardal, Lovasz, Lenstra). Minimum size trees?
- ▶ Greedy heuristic by Owen ahd Mehrotra, 2001
- ▶ Local branching heuristic for feasibility (Fischetti and Lodi, 2003)
- ▶ Heuristics based on Gomory Cuts (Karamanov and Cornuéjols, 2007, Cornuéjols, 2008)

Branching on General Disjunctions

$$\text{MIP:} \quad \begin{array}{rcl} z^* & = & \min cx \\ \text{s.t. } Ax & \geq & b \\ x & \in & \mathbb{Z}^n \end{array}$$

$$\text{LP:} \quad \begin{array}{rcl} z_{LP}^* & = & \min cx \\ \text{s.t. } Ax & \geq & b \\ x & \in & \mathbb{R}^n \end{array}$$

$$\begin{array}{rcl} z_L^* = \min cx & & \\ \text{s.t. } Ax & \geq & b \\ \pi x & \leq & \pi_0 \\ x & \in & \mathbb{R}^n \end{array} \qquad \begin{array}{rcl} z_R^* = \min cx & & \\ \text{s.t. } Ax & \geq & b \\ \pi x & \geq & \pi_0 + 1 \\ x & \in & \mathbb{R}^n \end{array}$$

New lower bound, $z_l = \min\{z_L^*, z_R^*\}$

Objective

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\}$ is maximized

Complexity

Optimization Problem

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\}$ is maximized

- ▶ For branching on variables, this can be solved in polynomial-time
- ▶ Is there a similar result when considering general disjunctions?
- ▶ When $(\pi, \pi_0) \in \{0, 1\}^{n+1}$?

Decision Problem

Given a MIP and $\hat{z}$, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\} > \hat{z}$

Decision Problem (2)

Given a MIP, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that LP relaxation of each branch is infeasible.

Complexity Results

Decision Problem (2)

Given a MIP, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that LP relaxation of each branch is infeasible.

Above problem is:

- $\mathcal{NP}-$complete in general (Reduction from Number Partitioning Problem)
- $\mathcal{NP}-$complete when $\pi \in \{0, 1\}^n$
- $\mathcal{NP}-$complete when $\pi \in \{-1, 0, 1\}^n$
- $\mathcal{NP}-$complete when $(\pi, \pi_0) \in \{0, 1\}^{n+1}$
- $\mathcal{NP}-$complete when $x \in \{0, 1\}^n$ and either of above four conditions hold (Reduction from 1-IN-3SAT)

Optimization Problem

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\}$ is maximized.
$\mathcal{NP}-$hard

Problem formulation: Maximum bound improvement

$$
\begin{aligned}
z_L^* = \min \; & cx \\
\text{s.t. } Ax \; & \geq \; b \\
\pi x \; & \leq \; \pi_0 \\
x \; & \in \; \mathbb{R}^n
\end{aligned}
\qquad\qquad
\begin{aligned}
z_R^* = \min \; & cx \\
\text{s.t. } Ax \; & \geq \; b \\
\pi x \; & \geq \; \pi_0 + 1 \\
x \; & \in \; \mathbb{R}^n
\end{aligned}
$$

Decision Problem

Given a MIP and $\hat{z}$, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\} > \hat{z}$?

Suppose $(\hat{\pi}, \hat{\pi}_0)$ is one such disjunction. Then the following systems must be infeasible

$$
\begin{aligned}
Ax \; & \geq \; b \\
cx \; & \leq \; \hat{z} \\
\hat{\pi} x \; & \leq \; \hat{\pi}_0 \\
x \; & \in \; \mathbb{R}^n
\end{aligned}
\qquad\qquad
\begin{aligned}
Ax \; & \geq \; b \\
cx \; & \leq \; \hat{z} \\
\hat{\pi} x \; & \geq \; \hat{\pi}_0 + 1 \\
x \; & \in \; \mathbb{R}^n
\end{aligned}
$$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$
\begin{array}{rcl}
Ax & \geq & b \\
cx & \leq & \hat{z} \\
\hat{\pi}x & \leq & \hat{\pi}_0 \\
x & \in & \mathbb{R}^n
\end{array}
\qquad \& \qquad
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Ax & \geq & b \\
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\end{array}
\qquad
\begin{array}{l}
\text{must be} \\
\text{infeasible}
\end{array}
$$

$$
\begin{array}{rcl}
z_1^* & = & \min \hat{\pi}x \\
Ax & \geq & b \\
cx & \leq & \hat{z} \\
x & \in & \mathbb{R}^n
\end{array}
\qquad \& \qquad
\begin{array}{rcl}
z_2^* & = & \min -\hat{\pi}x \\
Ax & \geq & b \\
cx & \leq & \hat{z} \\
x & \in & \mathbb{R}^n
\end{array}
\qquad
\begin{array}{l}
\text{must have} \\
z_1^* > \hat{\pi}_0, \\
z_2^* > -\hat{\pi}_0 - 1
\end{array}
$$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

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\begin{array}{rcl}
Ax & \geq & b \\
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$$

$$
\begin{array}{rcl}
z_1^* = \max pb & - & s_L z_l \\
pA - s_L c & = & -\hat{\pi} \\
p, s_L & \geq & 0
\end{array}
\qquad \& \qquad
\begin{array}{rcl}
z_2^* = \max qb & - & s_R z_l \\
qA - s_R c & = & -\hat{\pi} \\
q, s_R & \geq & 0
\end{array}
\qquad \text{(as above)}
$$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

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\begin{array}{rcl}
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$$
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z_1^* = \max pb & - & s_L z_l \\
pA - s_L c & = & -\hat{\pi} \\
p, s_L & \geq & 0
\end{array}
\quad \& \quad
\begin{array}{rcl}
z_2^* = \max qb & - & s_R z_l \\
qA - s_R c & = & -\hat{\pi} \\
q, s_R & \geq & 0
\end{array}
\quad \text{(as above)}
$$

$$
\begin{array}{rcl}
pb - s_L z_l & > & \hat{\pi}_0 \\
pA - s_L c & = & -\hat{\pi} \\
p, s_L & \geq & 0
\end{array}
\quad \& \quad
\begin{array}{rcl}
qb - s_R z_l & > & -\hat{\pi}_0 - 1 \\
qA - s_R c & = & -\hat{\pi} \\
q, s_R & \geq & 0
\end{array}
\quad
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Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$
\begin{aligned}
z_1^* = \max pb &- s_L z_l & & & z_2^* = \max qb &- s_R z_l & \\
pA - s_L c &= -\hat{\pi} & \& & & qA - s_R c &= -\hat{\pi} & \text{(as above)} \\
p, s_L &\geq 0 & & & q, s_R &\geq 0 &
\end{aligned}
$$

$$
\begin{aligned}
pb - s_L z_l &> \hat{\pi}_0 & & & qb - s_R z_l &> -\hat{\pi}_0 - 1 & \text{must be} \\
pA - s_L c &= -\hat{\pi} & \& & & qA - s_R c &= -\hat{\pi} & \text{feasible} \\
p, s_L &\geq 0 & & & q, s_R &\geq 0 &
\end{aligned}
$$

$$
\begin{aligned}
pb - s_L z_l &> \hat{\pi}_0 \\
pA - s_L c &= -\hat{\pi} \\
qb - s_R z_l &> -\hat{\pi}_0 - 1 \\
qA - s_R c &= -\hat{\pi} \\
q, s_R, p, s_L &\geq 0
\end{aligned}
$$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$
\begin{array}{llll}
z_1^* = \max pb & - & s_L z_l & \\
pA - s_L c & = & -\hat{\pi} & \& \\
p, s_L & \geq & 0 &
\end{array}
\qquad
\begin{array}{lll}
z_2^* = \max qb & - & s_R z_l \\
qA - s_R c & = & -\hat{\pi} \\
q, s_R & \geq & 0
\end{array}
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qA - s_R c & = & -\hat{\pi} \\
q, s_R, p, s_L & \geq & 0
\end{array}
\qquad
\begin{array}{lll}
pb - s_L z_l & > & \pi_0 \\
pA - s_L c & = & -\pi \\
qb - s_R z_l & > & -\pi_0 - 1 \\
qA - s_R c & = & -\pi \\
q, s_R, p, s_L & \geq & 0 \\
(\pi, \pi_0) & \in & \mathbb{Z}^n
\end{array}
$$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$
\begin{array}{rcl}
Ax & \geq & b \\
cx & \leq & \hat{z} \\
\hat{\pi}x & \leq & \hat{\pi}_0 \\
x & \in & \mathbb{R}^n
\end{array}
\qquad \& \qquad
\begin{array}{rcl}
Ax & \geq & b \\
cx & \leq & \hat{z} \\
\hat{\pi}x & \geq & \hat{\pi}_0 + 1 \\
x & \in & \mathbb{R}^n
\end{array}
\qquad
\begin{array}{l}
\text{must be} \\
\text{infeasible}
\end{array}
$$

$$
\begin{array}{rcl}
pb - s_L z_l & > & \hat{\pi}_0 \\
pA - s_L c & = & -\hat{\pi} \\
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q, s_R, p, s_L & \geq & 0
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\begin{array}{rcl}
pb - s_L z_l & > & \pi_0 \\
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qb - s_R z_l & > & -\pi_0 - 1 \\
qA - s_R c & = & -\pi \\
q, s_R, p, s_L & \geq & 0 \\
(\pi, \pi_0) & \in & \mathbb{Z}^n
\end{array}
$$

$(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction $\iff$ it satisfies the last MIP.

Problem formulation: Maximum bound improvement

$$
\begin{aligned}
pb - s_L z_l &> \pi_0 \\
pA - s_L c &= -\pi \\
qb - s_R z_l &> -\pi_0 - 1 \\
qA - s_R c &= -\pi \\
q, s_R, p, s_L &\geq 0 \\
(\pi, \pi_0) &\in \mathbb{Z}^n
\end{aligned}
$$

- $2n + 2$ constraints, $2m + n + 3$ variables.
- Tells us if we can increase the bound to z_l by branching at the current node
- Does not tell us the maximum value of z_l.
- We do a binary search over a range of z_l and solve the above MIP in each iteration

Problem formulation: Thin Directions

Similar approach has been used before:

- ▶ Cut generating LPs (Balas)
- ▶ Separation of a given point from the *Split Closure* (Caprara, 2003, Balas 2007)
- ▶ Finding thinnest directions?

Problem formulation: Thin Directions

Similar approach has been used before:

- ▶ Cut generating LPs (Balas)
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Thin Directions

- ▶ Thickness of the LP relaxation polytope $\mathcal{P}$ along a direction $\hat{\pi}$ is:

$$\max \hat{\pi}y - \hat{\pi}x, \quad x, y \in \mathcal{P}$$

- ▶ If the width of $\mathcal{P}$, $w(\mathcal{P})$ is minimum such width over all $\hat{\pi}$:

$$w(Q) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi y - \pi x) \ \ s.t. \ \ \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

Problem formulation: Thin Directions

$$w(Q) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi y - \pi x) \ \ s.t. \ \ \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

For a fixed $(\hat{\pi}, \hat{\pi}_0)$, the inner LP is:

$$\max \hat{\pi} y - \hat{\pi} x$$
$$\text{s.t. } Ax \geq b$$
$$Ay \geq b$$

Dual of the inner LP is:

$$\min -qb - pb$$
$$\text{s.t. } pA - \hat{\pi} = 0$$
$$qA + \hat{\pi} = 0$$
$$p, q \geq 0$$

Over all π:

$$\min -qb - pb$$
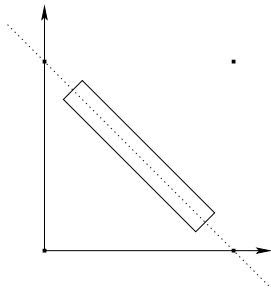$$pA - \pi = 0$$
$$qA + \pi = 0$$
$$p, q \geq 0$$
$$\pi \in \mathbb{Z}^n \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

Optional constraint:

$$\pi_0 < \pi x^* < \pi_0 + 1, \quad \pi_0 \in \mathbb{Z}$$

More on Thin Directions

- ▶ Problem of finding the thinnest direction is *NP*−hard. Remains hard even if the given LP polytope is a simplex (Sebö, 1999)
- ▶ Branching on thinnest direction does not necessarily result in the smallest tree (Examples by Krishnamoorthy, 2008).
- ▶ Branching on the thinnest direction need not make the child nodes infeasible.
- ▶ If a general branching disjunction makes the children infeasible then $w(\mathcal{P}) < 1$. Converse is not true.

Computational Experiments

- ► Objective: Study the effect of branching on *best* general hyperplanes on the size of branch-and-bound tree.
- ► We ignore time. Time spent on branching decisions $>>$ Time to solve the instance.
- ► Used ILOG CPLEX-10.2 MIP solver.
- ► Compared against strong branching (our own callback). Variable selected on the basis of maximum lower bound.
- ► Disabled Preprocessing, Heuristics.
- ► Cuts disabled in initial experiments.
- ► Provided best known upper bound to the solver.
- ► To perform experiments in reasonable time:
  - ► Imposed a limit of $-M \leq \pi_i \leq M$ (M is a fixed parameter).
  - ► Imposed a constraint of $\sum_i |\pi_i| \leq k$ (k is a fixed parameter).
  - ► Each MIP solved for finding a branching hyperplane had time limit t seconds (t is a fixed parameter).
  - ► Limit time on branching on a node to $8T$ seconds
  - ► If more than 18 hours spent on branching, then switch to strong branching on variables.
  - ► Total time limit of 20 hours for each instance.

Instances

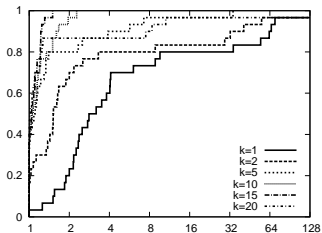30 Instances chosen from MIPLIB-3, MIPLIB-2003 and
Mittlemann's collection.

Instance	Cons	Vars	Ints	Bins	Instance	Cons	Vars	Ints	Bins
10teams	231	2025	1800	1800	mod008	6	319	319	319
aflow30a	479	842	421	421	neos6	1037	8768	8340	8340
bell3a	123	133	71	39	nug08	913	1632	1632	0
blend2	274	353	264	231	nw04	36	87482	87482	87482
egout	98	141	55	55	p0548	176	548	548	548
fiber	363	1298	1254	1254	pp08aCUTS	246	240	64	64
flugpl	18	18	11	0	qnet1	503	1541	1417	1288
gen	780	870	150	144	qnet1_o	456	1541	1417	1288
gesa2	1392	1224	408	240	ran10x26	297	520	260	260
gesa2_o	1248	1152	672	336	ran12x21	286	504	502	502
gt2	29	188	24	0	ran13x13	196	338	169	169
harp2	112	2993	2993	2993	rout	291	556	315	300
khb05250	101	1350	24	24	stein45	331	45	45	45
l152lav	97	1989	1989	1989	vpm1	234	378	168	168
lseu	28	89	89	89	vpm2	234	378	168	168

First Experiment

Instance	N_1	N_5	r_5	N_{10}	r_{10}	N_{15}	r_{15}	N_{20}	r_{20}
10teams	115	28	4.11	18	6.39	12	9.58	12	9.58
aflow30a	36634	19485	1.88	20388	1.8	24112	1.52	20271	1.81
bell3a	16387	8771	1.87	588	27.87	259	63.27	259	63.27
blend2	304	231	1.32	188	1.62	165	1.84	209	1.45
egout	2246	554	4.05	572	3.93	676	3.32	558	4.03
fiber	18412	7612	2.42	3039	6.06	3358	5.48	3324	5.54
flugpl	394	6	65.67	10	39.4	6	65.67	6	65.67
gen	100	100	1	100	1	100	1	100	1
gesa2	33526	21664	1.55	21849	1.53	21849	1.53	21778	1.54
gesa2_o	98550	24435	4.03	24661	4	24661	4	24661	4
gt2	340	10	34	12	28.33	10	34	12	28.33
harp2	432010	174656	2.47	183306	2.36	174454	2.48	179130	2.41
khb05250	738	594	1.24	588	1.26	614	1.2	618	1.19
l152lav	60	32	1.88	28	2.14	34	1.76	30	2
lseu	4058	226	17.96	78	52.03	58	69.97	58	69.97
mod008	2840	296	9.59	102	27.84	68	41.76	52	54.62
neos6	5989	2131	2.81	2131	2.81	2131	2.81	2131	2.81
nug08	14	4	3.5	6	2.33	6	2.33	5	2.8
nw04	30	16	1.88	12	2.5	12	2.5	12	2.5
p0548	1050	466	2.25	566	1.86	565	1.86	565	1.86
pp08aCUTS	1301300	147271	8.84	166943	7.79	168905	7.7	231527	5.62
qnet1	42	24	1.75	20	2.1	22	1.91	18	2.33
qnet1_o	154	94	1.64	77	2	80	1.93	92	1.67
ran10x26	68449	23309	2.94	24716	2.77	23704	2.89	21520	3.18
ran12x21	494558	219967	2.25	208948	2.37	225980	2.19	212910	2.32
ran13x13	124716	74699	1.67	57825	2.16	66008	1.89	58789	2.12
rout	219322	65201	3.36	61806	3.55	61226	3.58	57673	3.8
stein45	31086	21238	1.46	20594	1.51	20601	1.51	20601	1.51
vpm1	263111	145	1814.56	32	8222.22	20	13155.55	5929	44.38
vpm2	273994	77504	3.54	67014	4.09	69515	3.94	73687	3.72

Number of nodes (N_k) in tree and the ratio $r_k = \frac{N_1}{N_k}$ for selected instances when $t = 1000s$, $M = 1$, k is varied.
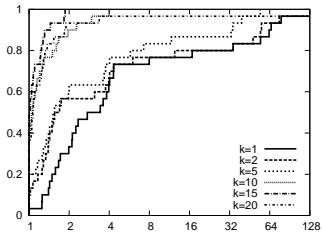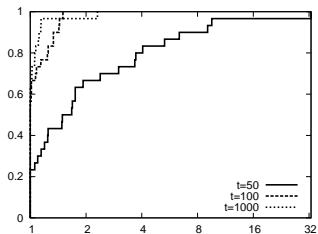
*Using performance profiles**
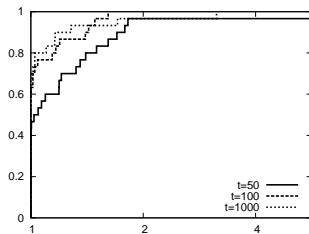

(a) $t = 1000$ seconds

(b) $t = 100$ seconds

(c) $t = 50$ seconds

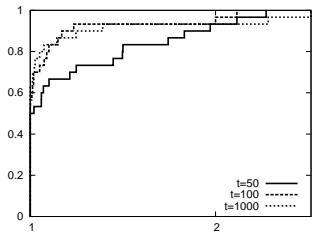Performance profile for number of subproblems when t is fixed and k is varied. $M = 1$

*Dolan, 2002

Dependence on t
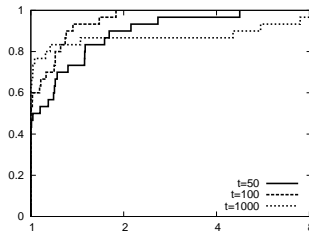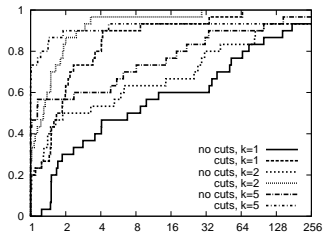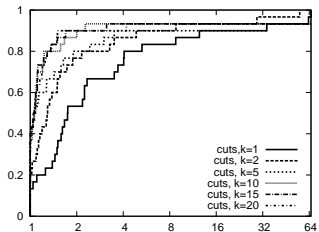

(d) $k = 5$

(e) $k = 10$

(f) $k = 15$

(g) $k = 20$

Performance profile for number of subproblems when k is fixed and t (in seconds) is varied.
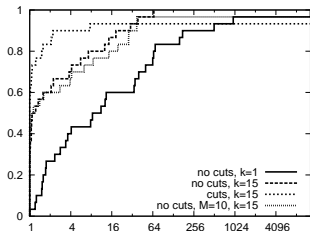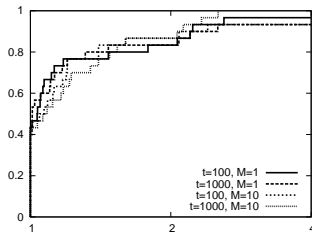
In presence of cuts

(h) $t = 100$ seconds   (i) $t = 100$ seconds

Figure: Performance profile for number of subproblems when cuts are added to the original problem. t is fixed and k is varied.
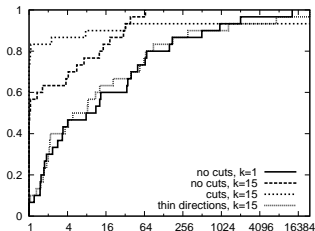
Having higher coefficients and thin directions

M is increased to 10, $t = 100s$

Effects of changing t when M in increased, $t = 100$ seconds

Comparison of effect of branching on *thin* directions against other criteria, $t = 100$ seconds

Concluding remarks

- ► The problem of finding a general disjunction that maximizes the lower bound is formulated
- ► The problem is shown to be $\mathcal{NP}-$hard in the general case and also for several specific restrictions
- ► A MIP formulation that can be solved to find such disjunctions was shown
- ► Computational experiments show that these disjunctions can reduce the size of branch-and-bound tree by a significant amount.
- ► The time taken to solve these MIPs is prohibitively large
- ► We are currently working on techniques to speed up this approach and develop heuristics

- ► A.M. and T.K.R., *Experiments with Branching using General Disjunctions*, Proceedings of the Eleventh INFORMS Computing Society Meeting, 2009, To Appear
- ► A.M. and T.K.R., *On the complexity of selecting branching disjunctions for integer programming*, Working Paper

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