Comparing Nonconvex Programs: Discrete and Continuous

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Overview

Introduction to RLT-based techniques

- A lift-and-project tightening for linear 0-1 programs
- Can I get the convex hull?

Quadratic Programs: Discrete (0-1) Variables

- Constructing a higher-dimensional LP representation
- Generating RLT constraints: Bound-factor products
- A branch-and-bound algorithm

Quadratic Programs: Continuous Variables

- Constructing a higher-dimensional LP representation
- A branch-and-bound algorithm
- Comparing the discrete and continuous cases

Decision Tree Analysis

- Path-based formulation
- Discrete and Continuous Representations

INTRODUCTION TO RLT:
A LINEAR 0-1 EXAMPLE

Solving MINLPs using LPs!

- 0-1 programs to continuous (nonconvex) factorable programs
- DC or DM optimization, branch-and-reduce, *etc*.
- BARON, OQNLP, LGO, *etc*.

***Reformulation – Linearization Technique* (RLT)**

Basic idea:

Easiest optimization problems to solve are linear programs (LPs)

Take advantage to construct tight higher-dimensional LP representations to a given nonconvex program

Derive other valid inequalities to strengthen the RLT relaxation

Embed the RLT relaxation into a (convergent) branch-and-bound process

Hmm…does RLT really help?

Consider the following linear 0-1 program:

Maximize $3x_1 + 4x_2$

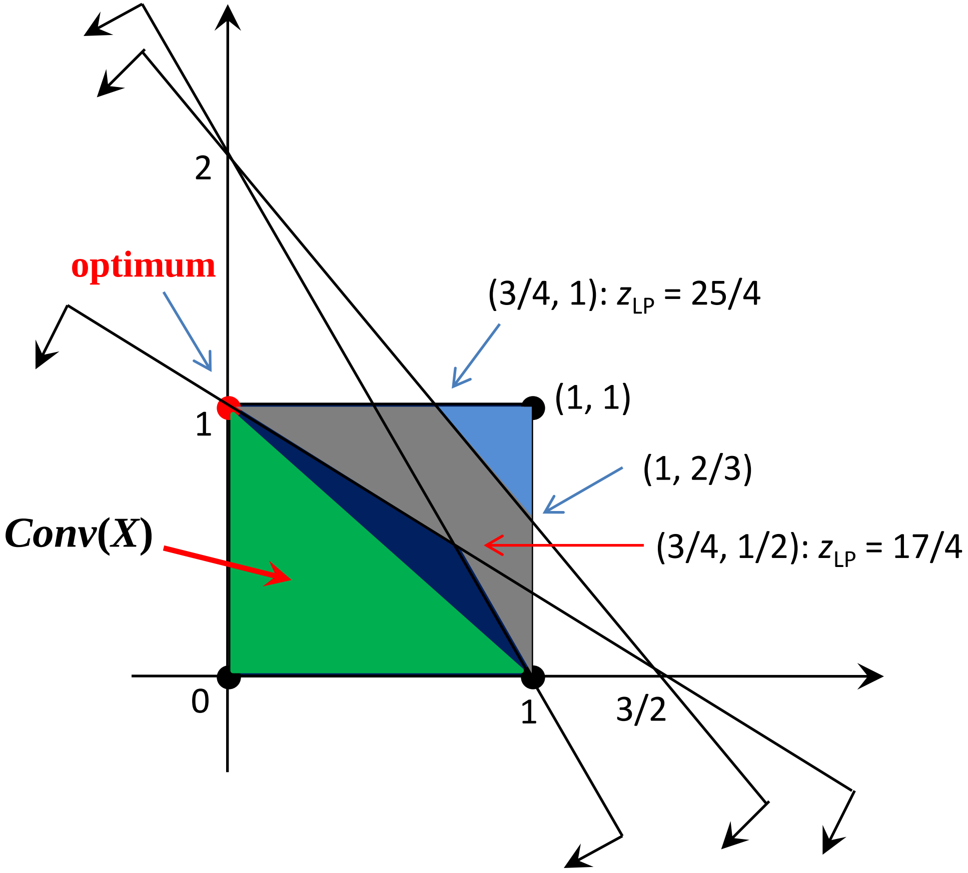
s.t: $4x_1 + 3x_2 \le 6$

$(x_1, x_2) \in \{0, 1\}^2$

$$4x_1 + 3x_2 \le 6 \begin{cases} \times (x_1 - 0) \\ \times (1 - x_1) \end{cases}$$

RLT constraints

$$\left\{ \begin{array}{l} -2x_1 + \quad + 3w_{12} \le 0 \\ \boxed{2x_1 + x_2 - w_{12} \le 2} \\ -3x_2 + 4w_{12} \le 0 \\ \boxed{2x_1 + 3x_2 - 2w_{12} \le 2} \end{array} \right.$$



Hmm…does RLT really help? (contd.)

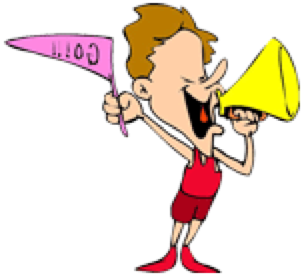
How about going one step further?

$$2x_1 + 3x_2 \le 3 \quad \leftarrow \quad \times(1-x_1)$$

$$\Rightarrow \quad 3x_2 - 3w_{12} \le 3 - 3x_1$$

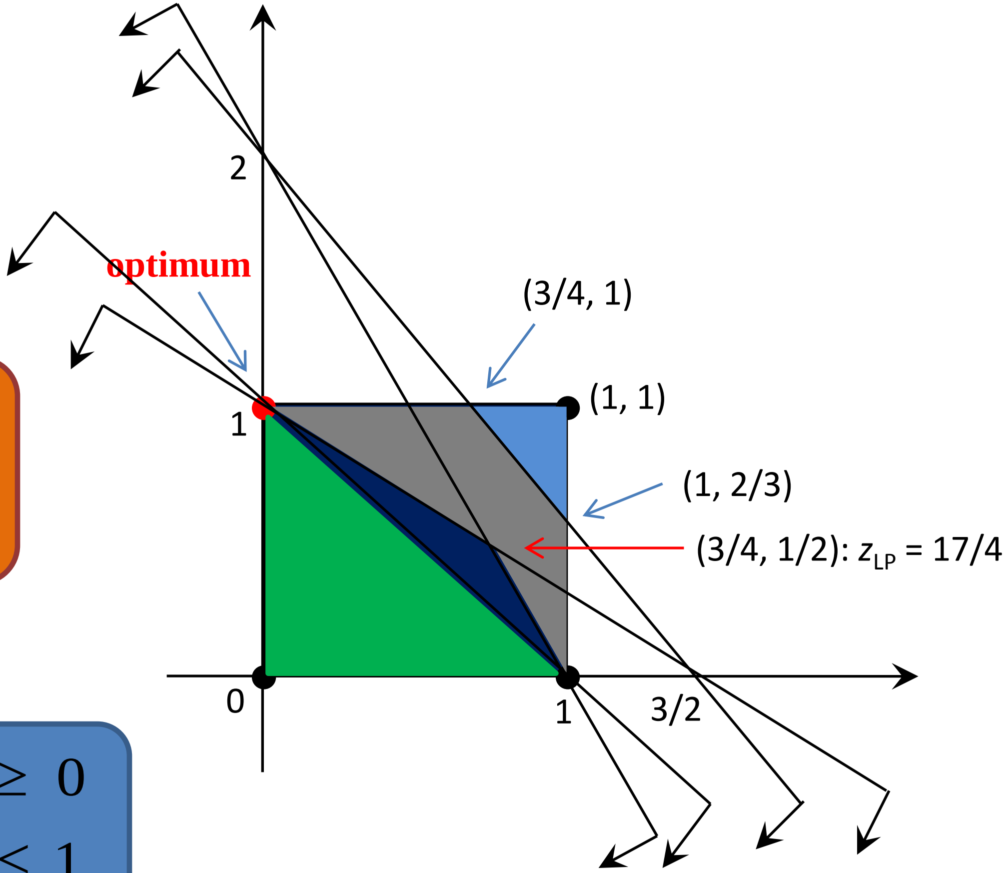
$$\Rightarrow \quad \boxed{x_1 + x_2 - w_{12} \le 1}$$

We have generated the Convex Hull ☺!



$$(1-x_1)(1-x_2) \ge 0$$

$$\Rightarrow \quad x_1 + x_2 - w_{12} \le 1$$



QUADRATIC PROGRAMS: DISCRETE (0 – 1) VARIABLES

Pure 0-1 QPs

$$\text{Minimize} \quad \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij} \boxed{x_i x_j} \quad \Longrightarrow \quad \underline{y_{ij} = x_i x_j}$$

RLT VARIABLES

$$\text{subject to} \quad \sum_{i=1}^{n} a_{ki} x_i \le b_k , \ \forall\, k \quad \rightarrow \quad x_j, \quad (1 - x_j)$$

$$x_i \in \{0, 1\}$$

RLT CONSTRAINTS

$$\begin{cases} y_{ij} \le x_i, & \forall\, i \\ y_{ij} \le x_j, & \forall\, j \\ -y_{ij} + x_i + x_j \le 1, & \forall\, (i, j) \\ y \ge 0 & \end{cases}$$

LP relaxation: $(\bar{x}, \bar{y})$; *Branching Strategy:* $\theta_{uv} = \arg\max_{(i,j)} \left\{ \left| \bar{y}_{ij} - \bar{x}_i \bar{x}_j \right| \right\}$

Bound-Factor Products

$$\begin{tikzcd}
(x_i - 0) \arrow[loop left] \arrow[r] \arrow[d] \arrow[dr] & (1 - x_i) \arrow[loop right] \arrow[d] \arrow[dl] \\
(x_j - 0) \arrow[loop left] \arrow[r] & (1 - x_j) \arrow[loop right]
\end{tikzcd}$$

$$e.g. \quad x_i(1 - x_j) \geq 0$$

$$\Rightarrow \quad x_i - x_i x_j \geq 0 \quad \Rightarrow \quad x_i - y_{ij} \geq 0.$$

$$e.g. \quad (1 - x_i)(1 - x_j) \geq 0$$

$$\Rightarrow \quad 1 - x_i - x_j + x_i x_j \geq 0$$

$$\Rightarrow \quad -x_i - x_j + y_{ij} \geq -1.$$

Pure 0-1 QPs (contd.)

Minimize $f(x_1,x_2) = x_1 + x_1^2 - 2x_2 - x_2^2 - 5\overset{w_{12}}{x_1 x_2}$

subject to:

$$\left.\begin{aligned} 4x_1 + 2x_2 &\le 5 \\ -2x_1 + 5x_2 &\le 4 \end{aligned}\right\} \begin{array}{l} \nearrow \; x_1;\ x_2 \\ \searrow \; (1-x_1);\ (1-x_2) \end{array}$$

$$x_1, x_2 \in \{0,1\}.$$

$$\begin{aligned} e.g.\quad & (4x_1 + 2x_2 - 5)(1 - x_2) \le 0 \\ \Rightarrow\quad & 4x_1 + 2x_2 - 5 \ - x_2(4x_1 + 2x_2 - 5) \le 0 \\ \Rightarrow\quad & 4x_1 + 2x_2 - 5 \ - \ 4w_{12} - 2x_2 + 5x_2 \ \le \ 0 \\ \Rightarrow\quad & 4x_1 + \ 5x_2 - 4w_{12} \ \le \ 5 \end{aligned}$$

RLT Relaxation

Minimize $f(x_1, x_2, w_{12}) = 2x_1 - 3x_2 - 5w_{12}$

subject to:

$$\left.\begin{aligned} -x_1 + 2w_{12} &\le 0 \\ -3x_2 + 4w_{12} &\le 0 \\ -6x_1 + 5w_{12} &\le 0 \\ x_2 - 2w_{12} &\le 0 \end{aligned}\right\} \textbf{Type 1 constraints}$$

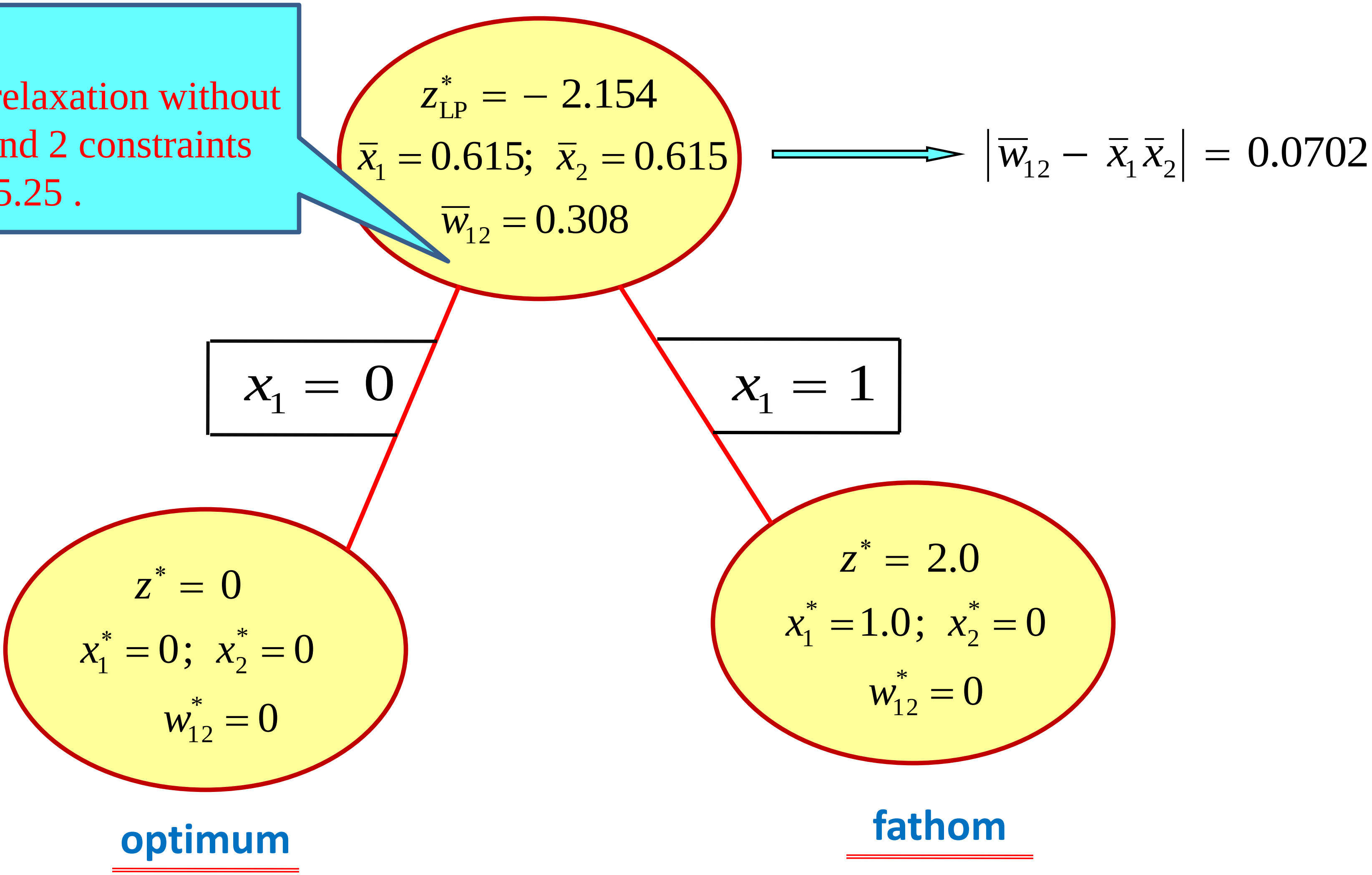
$$\left.\begin{aligned} 5x_1 + 2x_2 - 2w_{12} &\le 5 \\ 4x_1 + 5x_2 - 4w_{12} &\le 5 \\ 4x_1 + 5x_2 - 5w_{12} &\le 4 \\ -2x_1 + 4x_2 + 2w_{12} &\le 4 \end{aligned}\right\} \textbf{Type 2 constraints}$$

$$\left.\begin{aligned} w_{12} &\le x_1 \\ w_{12} &\le x_2 \\ -w_{12} + x_1 + x_2 &\le 1 \end{aligned}\right\} \textbf{Type 3 constraints}$$

$$x_1, x_2 \in \{0, 1\}; \;\; w_{12} \ge 0.$$

Branch-and-Bound Tree

Remark:
The LP relaxation without Type 1 and 2 constraints is $z^* = -5.25$.



$z^*_{\mathrm{LP}} = -2.154$

$\bar{x}_1 = 0.615;\ \bar{x}_2 = 0.615$

$\bar{w}_{12} = 0.308$

$\left|\bar{w}_{12} - \bar{x}_1 \bar{x}_2\right| = 0.0702$

$x_1 = 0$

$x_1 = 1$

$z^* = 0$

$x_1^* = 0;\ x_2^* = 0$

$w_{12}^* = 0$

optimum

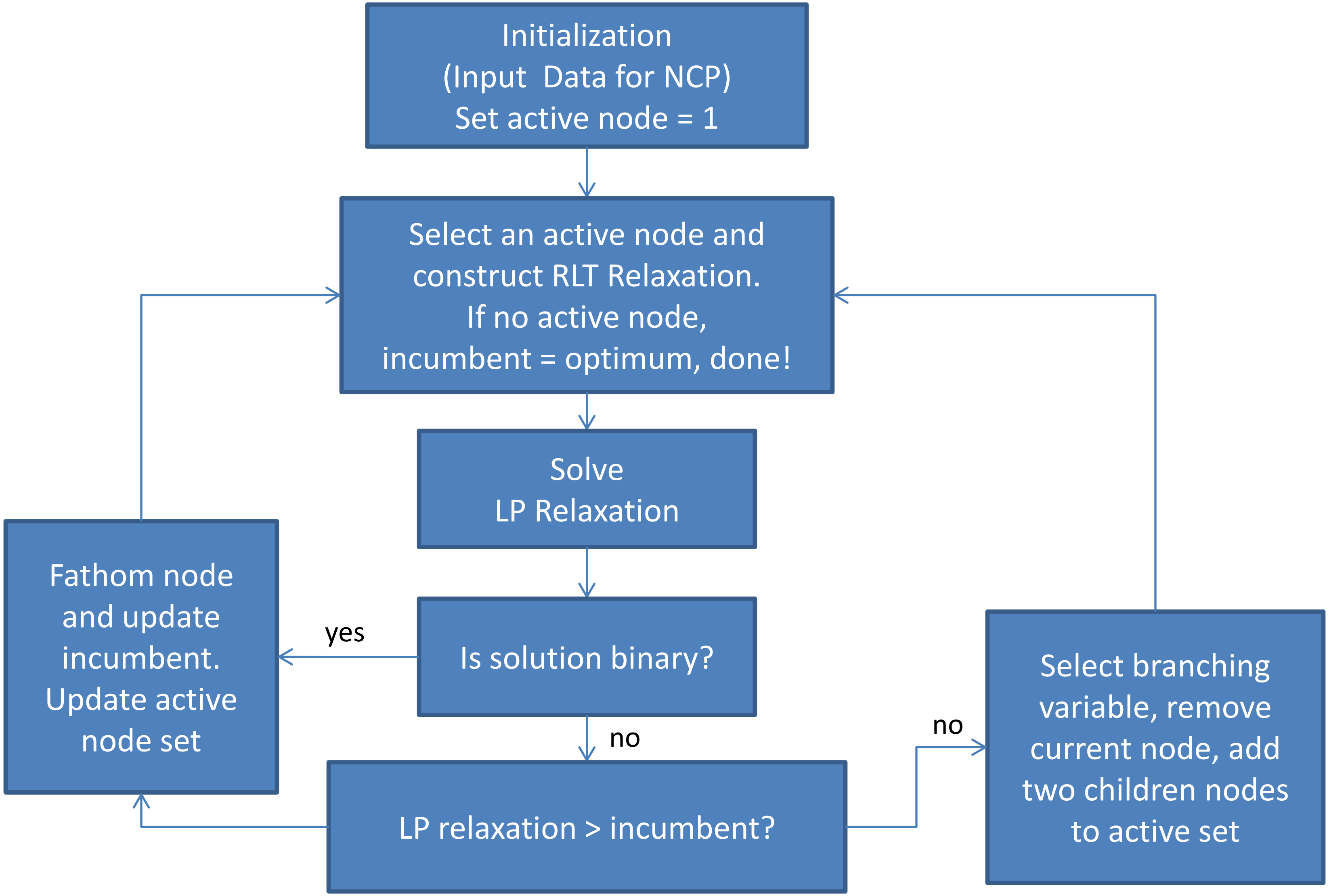
$z^* = 2.0$

$x_1^* = 1.0;\ x_2^* = 0$

$w_{12}^* = 0$

fathom

Branch-and-Bound Algorithm



Initialization
(Input Data for NCP)
Set active node = 1

Select an active node and
construct RLT Relaxation.
If no active node,
incumbent = optimum, done!

Solve
LP Relaxation

Is solution binary?

yes

Fathom node
and update
incumbent.
Update active
node set

no

LP relaxation > incumbent?

no

Select branching
variable, remove
current node, add
two children nodes
to active set

QUADRATIC PROGRAMS: CONTINUOUS VARIABLES

Continuous Nonconvex Programs

Minimize $f(x_1, x_2) = x_1 + x_1^2 - 2x_2 - x_2^2 - 5x_1x_2$

subject to:

$$4x_1 + 2x_2 \le 5$$

$$-2x_1 + 5x_2 \le 4$$

$$0 \le x_1, x_2 \le 1.$$

$$e.g. \quad (1 - x_1)(1 - x_2) \ge 0$$

$$\Rightarrow \quad -x_1 - x_2 + w_{12} \le 1$$

$$e.g. \quad (4x_1 + 2x_2 - 5)(1 - x_2) \le 0$$

$$\Rightarrow \quad 4x_1 + 2x_2 - 5 - 4x_1x_2 - 2x_2^2 + 5x_2 \le 0$$

$$\Rightarrow \quad 4x_1 + 7x_2 - 4w_{12} - 2w_{22} \le 5$$

RLT Relaxation (Shortened)

Minimize $f(x_1, x_2) = x_1 + w_{11} - 2x_2 - w_{22} - 5w_{12}$

subject to:

$$4x_1 + 2x_2 \leq 5$$

$$-2x_1 + 5x_2 \leq 4$$

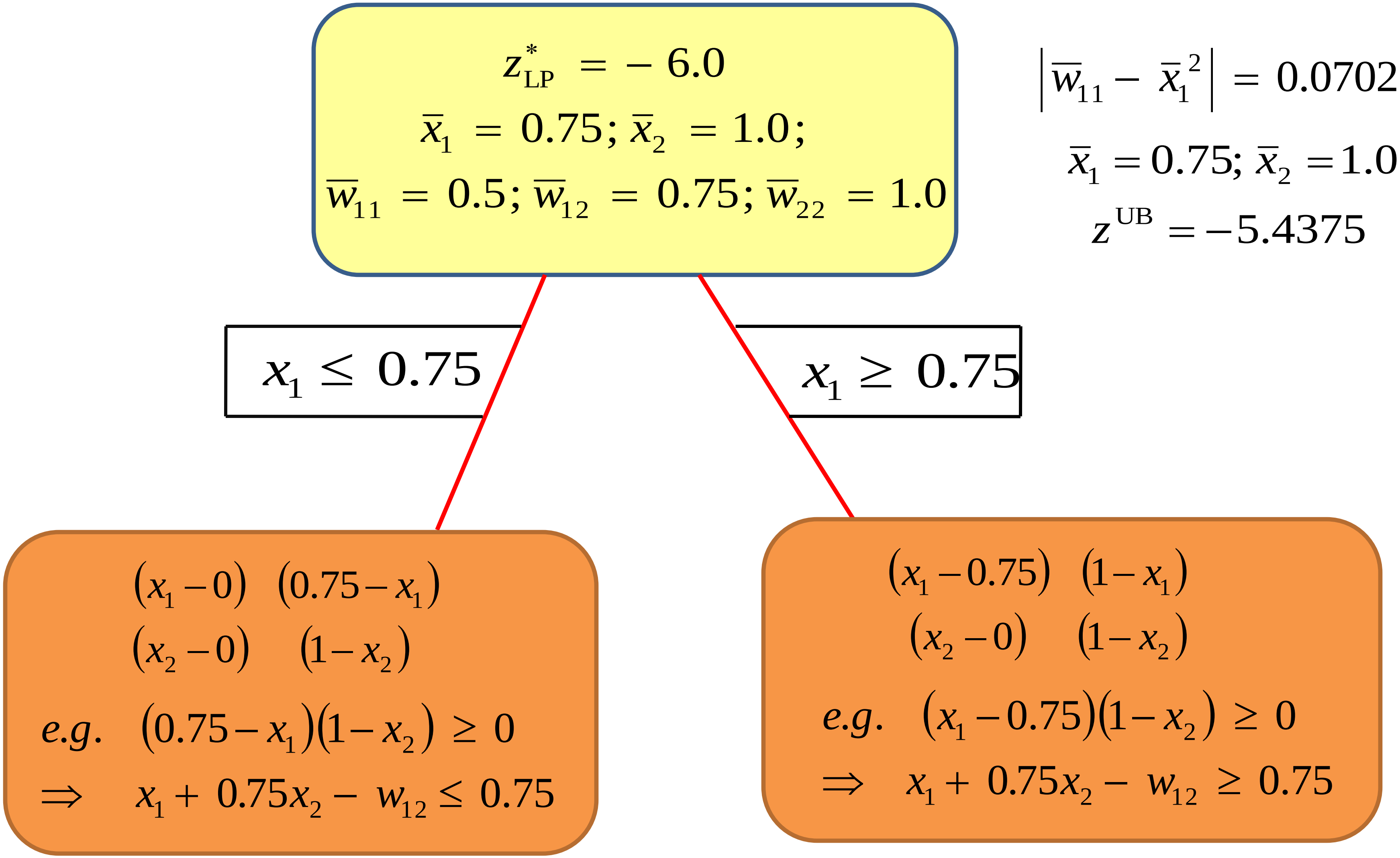
BOUND-FACTOR PRODUCTS

$$\left\{\begin{array}{r}
x_1 \leq w_{11} \\
x_1 \leq w_{12} \\
x_2 \leq w_{12} \\
x_2 \leq w_{22} \\
x_1 - w_{11} \leq 1 \\
x_2 - w_{22} \leq 1 \\
x_1 + x_2 - w_{12} \leq 1
\end{array}\right.$$

$$0 \leq x_1, x_2, x_3, w_{11}, w_{12}, w_{22} \leq 1$$

CONSTRAINT-BOUND FACTOR PRODUCTS NOT INCLUDED IN THIS RELAXATION !

Branch-and-Bound Tree



$$z^{*}_{\mathrm{LP}} = -6.0$$
$$\bar{x}_1 = 0.75;\ \bar{x}_2 = 1.0;$$
$$\bar{w}_{11} = 0.5;\ \bar{w}_{12} = 0.75;\ \bar{w}_{22} = 1.0$$

$$\left|\bar{w}_{11} - \bar{x}_1^{\,2}\right| = 0.0702$$
$$\bar{x}_1 = 0.75;\ \bar{x}_2 = 1.0$$
$$z^{\mathrm{UB}} = -5.4375$$

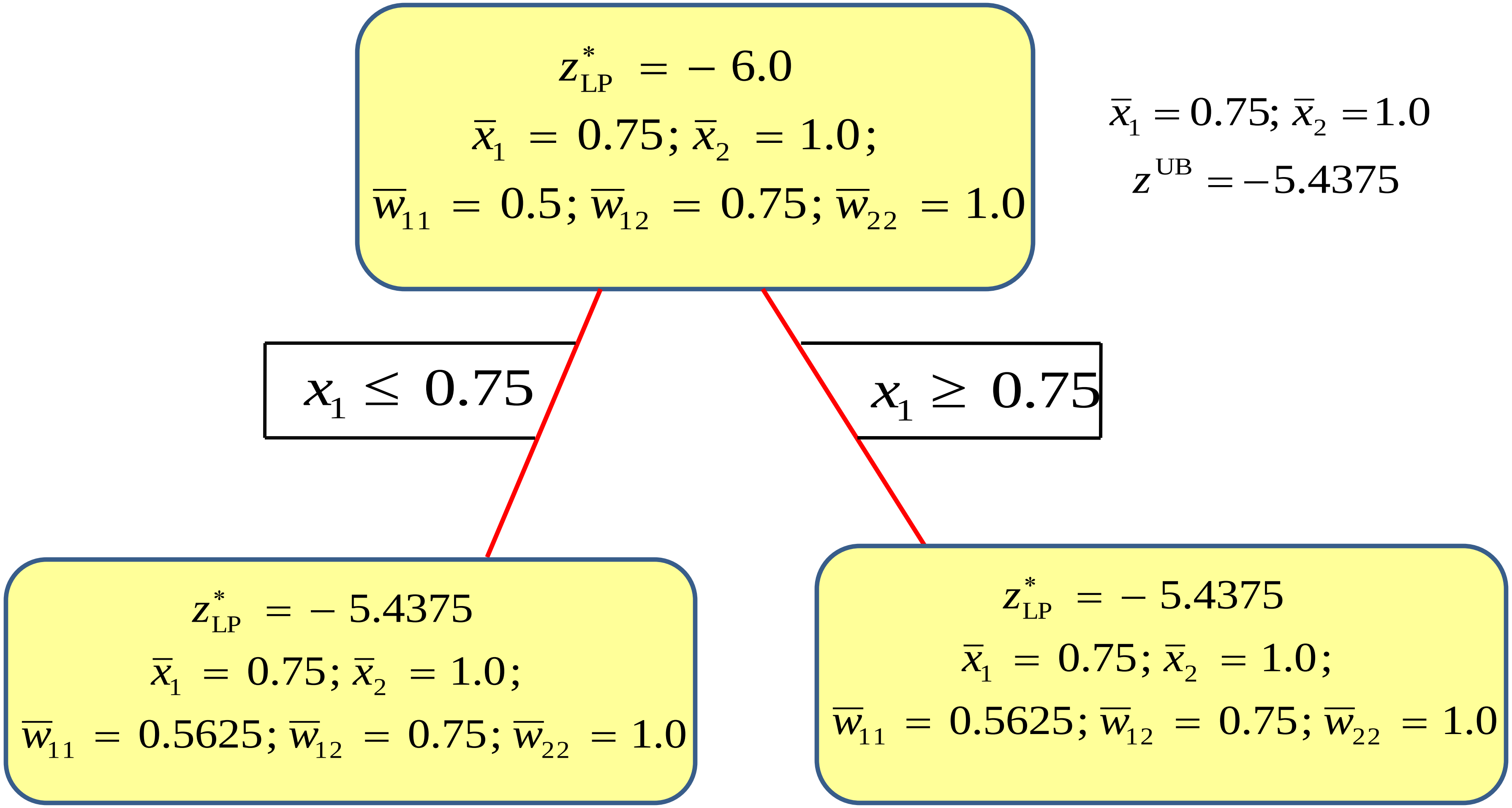
$x_1 \leq 0.75$

$x_1 \geq 0.75$

$$(x_1 - 0) \quad (0.75 - x_1)$$
$$(x_2 - 0) \quad (1 - x_2)$$
$$e.g. \quad (0.75 - x_1)(1 - x_2) \geq 0$$
$$\Rightarrow \quad x_1 + 0.75x_2 - w_{12} \leq 0.75$$

$$(x_1 - 0.75) \quad (1 - x_1)$$
$$(x_2 - 0) \quad (1 - x_2)$$
$$e.g. \quad (x_1 - 0.75)(1 - x_2) \geq 0$$
$$\Rightarrow \quad x_1 + 0.75x_2 - w_{12} \geq 0.75$$

Branch-and-Bound Tree (contd.)



$z^*_{\mathrm{LP}} = -6.0$

$\bar{x}_1 = 0.75; \bar{x}_2 = 1.0;$

$\overline{w}_{11} = 0.5; \overline{w}_{12} = 0.75; \overline{w}_{22} = 1.0$

$\bar{x}_1 = 0.75; \bar{x}_2 = 1.0$

$z^{\mathrm{UB}} = -5.4375$

$x_1 \leq 0.75$

$x_1 \geq 0.75$

$z^*_{\mathrm{LP}} = -5.4375$

$\bar{x}_1 = 0.75; \bar{x}_2 = 1.0;$

$\overline{w}_{11} = 0.5625; \overline{w}_{12} = 0.75; \overline{w}_{22} = 1.0$

$z^*_{\mathrm{LP}} = -5.4375$

$\bar{x}_1 = 0.75; \bar{x}_2 = 1.0;$

$\overline{w}_{11} = 0.5625; \overline{w}_{12} = 0.75; \overline{w}_{22} = 1.0$

optimum

SOLVING DISCRETE QUADRATIC PROGRAMS USING CONTINUOUS VARIABLES

A Continuous Representation

Minimize $f(x_1, x_2) = x_1 + x_1^2 - 2x_2 - x_2^2 - 5x_1x_2$

subject to:

$$4x_1 + 2x_2 \le 5$$

$$-2x_1 + 5x_2 \le 4$$

$$x_1, x_2 \in \{0, 1\}.$$

$$f(x_1, x_2, w_{11}, w_{12}, w_{22}) = x_1 + w_{11} - 2x_2 - w_{22} - 5w_{12}$$

$$4x_1 + 2x_2 \le 5$$

$$-2x_1 + 5x_2 \le 4$$

$$x_1 - w_{11} = 0$$

$$x_2 - w_{22} = 0$$

$$w_{12} \le x_1$$

$$w_{12} \le x_2$$

$$-w_{12} + x_1 + x_2 \le 1$$

$$0 \le x_1, x_2, w_{11}, w_{12}, w_{22} \le 1.$$

A Continuous Representation (contd.)

$$z^*_{\mathrm{LP}} = -5.25$$
$$\bar{x}_1 = 0.75; \bar{x}_2 = 1.0;$$
$$\bar{w}_{11} = 0.75; \bar{w}_{12} = 0.75; \bar{w}_{22} = 1.0$$

$$z^*_{\mathrm{LP}} = -2.1538$$
$$\bar{x}_1 = 0.615; \bar{x}_2 = 0.615;$$
$$\bar{w}_{11} = 0.615; \bar{w}_{12} = 0.308; \bar{w}_{22} = 0.615$$

Summary

- Looked at constructing RLT relaxations for quadratic discrete (0-1) programs and quadratic programs defined in terms of continuous variables
- Branch-and-bound algorithms for both cases
- A continuous representation for quadratic 0-1 problems
- Comparison of relaxations
- Ideas for further talks include insights into developing relaxations for higher order polynomial programs, tightening the RLT relaxation using semidefinite cuts, and applications.

QUESTIONS?

THANKS. . .