Cutting planes from two rows of simplex tableau

Based on talk by Andersen et al, IPCO-2007

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COMPUTATIONAL OPTIMIZATION
RESEARCH AT LEHIGH

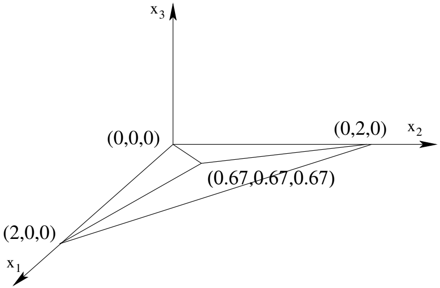
Split Cuts

- Subsume MIR, GMI, Lift and Project, ...
- Effectively used in branch-and-cut algorithms
- Let $(\pi, \pi_0) \in \mathbb{Z}^{(n+1)}$. Any valid inequality for $P \cap \{x | \pi x \leq \pi_0\}$ and $P \cap \{x | \pi x \geq \pi_0 + 1\}$ is valid for P
- Split cuts alone are NOT sufficient to solve a general MIP problem
- Need of stronger classes of cuts
- How to split on multiple disjunctions?
- Also see: *Mixing mixed integer inequalities* by Günluk and Pochet (and Cor@l talk by Kumar Abhishek)

Today's focus

Cutting planes from two rows of a simplex tableau
`http://www.math.uni-magdeburg.de/~louveaux/AndLouWeiWol-2may.pdf`

Cook, Kannan and Schrijver's example

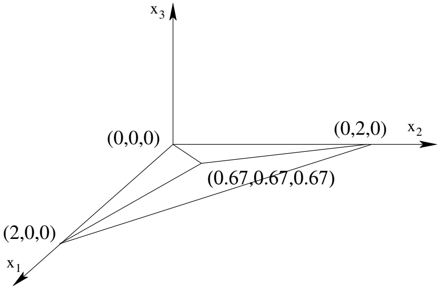


$$\begin{array}{rcl} \min -x_3 & & \\ \text{s.t.} & & \\ x_3 & \leq & x_1 \\ x_3 & \leq & x_2 \\ x_1 + x_2 + x_3 & \leq & 2 \\ x_1, x_2 & \in & \mathbb{Z} \\ x_3 & \in & \mathbb{R}^+ \end{array}$$

$x_3 \leq 0$ is a valid cut for the above polytope.

Proof: ...

Cook, Kannan and Schrijver's example



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$x_3 \leq 0$ is not a split inequality. See *Cook et al*.

The simplex tableau

- I: set of integer variables, C: set of continuous variables
- B: set of basic variables, N: set of non-basic variables
- rows in simplex tableau:

$$x_i = f_i + \sum_{j \in N} r^j x_j, \quad \forall i \in B$$

- If $f_i \in \mathbb{Z} \quad \forall i \in B \cap I$, current solution is feasible
- if $f_i \notin \mathbb{Z}$ then cuts may be derived from this row (MIR, GMI)
- Lets consider two rows (with change of notation) now:

$$\begin{aligned} x_1 &= f_1 + \sum_{j \in N} r_1^j s_j \\ x_2 &= f_2 + \sum_{j \in N} r_2^j s_j \end{aligned} \qquad \text{or} \qquad \mathbf{x} = \mathbf{f} + \sum_{j \in N} \mathbf{r}^j s_j$$

$\mathbf{x}, \mathbf{f}, \mathbf{r}^j$ are vectors in two dimensions

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First steps

$$P_I = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j\}$$

$$P_{LP} = \{(x,s) \in \mathbb{R}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j\}$$

r^j : also called a *ray*, as in an LP

P_I may be empty. (Never so for 1-row case.) e.g.

$$x_1 = \frac{1}{5} + 3s_1 + 4s_2$$

$$x_2 = \frac{2}{3} + 3s_1 + 4s_2$$

Lemma: P_I is empty if and only if

1. All rays $\{r^j\}$ are parallel, and
2. The lines $\{f + r^j s_j : s_j \in \mathbb{R}\}$ for $j \in N$ do not contain any integer points

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Structure of conv(P_I)

$$
\begin{aligned}
-5x_1 + 3x_2 &\leq -1 \\
x_1 - 5x_2 &\leq -2 \\
x_1, x_2 &\in \mathbb{Z}
\end{aligned}
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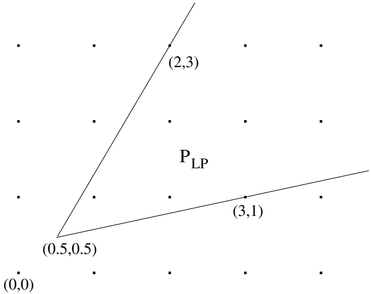

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-5x_1 + 3x_2 + s_1 &= -1 \\
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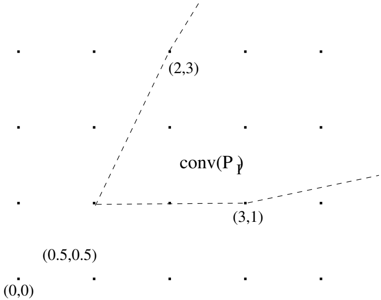

$conv(P_I)$ (Not empty)

Let $P_I = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j\}$

Lemma:

1. The extreme rays of $conv(P_I)$ are (r^j, e_j) for $j \in N$
2. The dimension of $conv(P_I)$ is $n(= |N|)$
3. The vertices (x^I, s^I) of $conv(P_I)$ take either of two forms:
   - 3.1 $(x^I, s^I) = (x^I, e_j s^I_j)$, where $x^I = f + r^j s^I_j \in \mathbb{Z}^2$ and $j \in N$. (integer point on ray $\{f + r^j s_j : s_j \geq 0)$
   - 3.2 $(x^I, s^I) = (x^I, e_j s^I_j + e_k s^I_k)$, where $x^I = f + r^j s^I_j + r^k s^I_k \in \mathbb{Z}^2$ and $j, k \in N$. (integer point in the set $f + cone(\{r^j, r^k\})$.

Proof: ... (Not all points satisfying above properties are vertices.)

Corollary: Every non-trivial valid inequality for P_I that is tight at a point $(\bar{x}, \bar{s}) \in P_I$ can be written in the form

$$\sum_{j \in N} \alpha_j s_j \geq 1,$$

where $\alpha_j \geq 0$ for all $j \in N$.

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Valid inequality for $conv(P_I)$

Let $\sum_{j\in N} \alpha_j s_j \geq 1$ be a valid inequality for $conv(P_I)$ that is tight for P_I. Let

$$L_\alpha = \{x \in \mathbb{R}^2 : \exists\, s \in \mathbb{R}^n_+ \;\; s.t. (x,s) \in P_{LP} \text{ and } \sum_{j\in n} \alpha_j s_j \leq 1\}$$

Lemma: Let $v^j = f + \frac{1}{\alpha_j} r^j, j \in N\backslash N^0_\alpha$, then

1. $interior(L_\alpha) \cap P_I = \phi$
2. if $interior(L_\alpha) \neq \phi$, then $f \in interior(L_\alpha)$
3. $L_\alpha = conv(\{f\} \cup \{v^j\}_{j\in N\backslash N^0_\alpha}) + cone(\{r^j\}_{j\in N^0_\alpha})\}$

Proof: ...

$$\begin{aligned}
\text{Let } X_\alpha &= \{x \in \mathbb{Z}^2 : \exists\, s \in \mathbb{R}^n_+ s.t. (x,s) \in P_{LP} \text{ and } \sum_{j\in N} \alpha_j s_j = 1\} \\
&= L_\alpha \cap \mathbb{Z}^2 \\
&\neq \phi \text{ when } \sum_{j\in N} \alpha_j s_j = 1 \text{ is a facet}
\end{aligned}$$

not neccessarily true for faces or other valid inequalities

Split cuts

Lemma: If, for a facet defining inequality $\sum_{j \in N} \alpha_j s_j \geq 1$ for $conv(P_I)$, $N_\alpha^0 \neq \phi$ then $\exists (\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$ $s.t.$ $L_\alpha \subseteq \{(x_1, x_2) : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$.

Proof:

1. Let $k \in N_\alpha^0$. Then the line $\{f + \mu r^k : \mu \in \mathbb{R}\}$ does not pass through any integer points in $\mathbb{R}^2$
2. All rays $\{r^j\}_{j \in N_\alpha^0}$ are parallel
3. Let $\pi' = (-r_2^k, r_1^k)$, $\pi_0' = \pi' f$. Then, $\{f + \mu r^k : \mu \in \mathbb{R}\} = \{x | \pi' x = \pi_0'\}$
4. Let

$$\pi_0^1 = \max\{\pi_1' x_1 | \pi_2' x_2 \leq \pi_0', x \in \mathbb{Z}^2\}$$
$$\pi_0^2 = \min\{\pi_1' x_1 | \pi_2' x_2 \leq \pi_0', x \in \mathbb{Z}^2\}$$
$$S_\pi = \{x \in \mathbb{R}^2 : \pi_0^1 \leq \pi_1' x_1 + \pi_2' x_2 \leq \pi_0^2\}$$

5. $L_\alpha \subseteq S_\pi$
6. $S_\pi = \{x \in \mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$ for some $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$.

When $N_{\alpha}^{0} = \phi$

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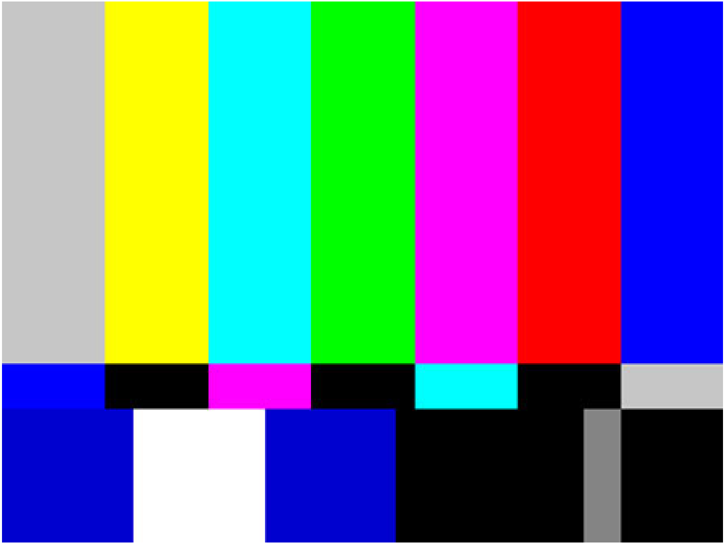

Stay tuned

Recap

$$P_I = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j\}$$

$$P_{LP} = \{(x,s) \in \mathbb{R}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j\}$$

- Basic solution: $(f, 0)$
- Objective: Find facet defining inequality(ies) for $conv(P_I)$.
- Some of these inequalities may not be split cuts (of any rank).
- P_{LP} is a cone.
- $dim(conv(P_I)) = n = |N|$
- For any $(\bar{x}, \bar{s}) \in P_I$, either:
  1. $\bar{x} = f + s_j e_j$ (ray point) or,
  2. $\bar{x} = f + s_j e_j + s_k e_k$

Recap

Every non-trivial valid inequality for P_I that is tight at a point $(\bar{x}, \bar{s}) \in P_I$ can be written in form:

$$\sum_{j \in N} \alpha_j s_j \geq 1,$$

where $\alpha_j \geq 0, \forall j \in N$.

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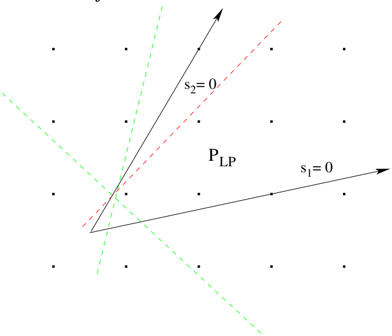
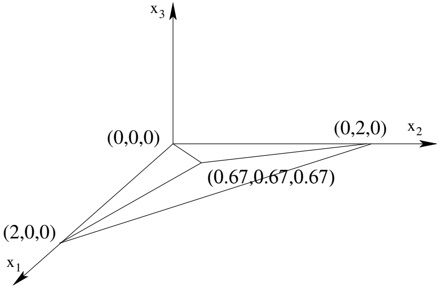


Example



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P_{LP} :

$$x_1 = \frac{2}{3} + \frac{2}{3}s_1 - \frac{1}{3}s_2 - \frac{1}{3}s_3$$

$$x_2 = \frac{2}{3} - \frac{1}{3}s_1 + \frac{2}{3}s_2 - \frac{1}{3}s_3$$

facet definining inequality: $x_3 \leq 0 \quad \Rightarrow \quad \frac{1}{2}s_1 + \frac{1}{2}s_2 + \frac{1}{2}s_3 \geq 1$

Split Cuts

Let $\sum_{j \in N} \alpha_j s_j \geq 0$ be a valid inequality for $conv(P_I)$. Then:

1. Let $N_\alpha^0 = \{j : \alpha_j = 0\}$,
2. $L_\alpha = \{x \in \mathbb{R}^n | \exists s \in \mathbb{R}^n_+ s.t. (x, s) \in P_{LP}$ and $\sum_{j \in N} \alpha_j s_j \leq 1\}$
3. $X_\alpha = L_\alpha \cap \mathbb{Z}^2$

Lemma: If, for a facet defining inequality $\sum_{j \in N} \alpha_j s_j \geq 1$ for $conv(P_I)$, $N_\alpha^0 \neq \phi$ then $\exists (\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$ $s.t.$
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e.g. previous example.

Converse is not true.

When $N_\alpha^0 \neq \phi$

Main results of this paper:

- Every facet is derivable from at most four non-basic variables
- With every facet, one can associate three or four particular vertices of $conv(P_I)$. These facets can be classified into:
  1. Split Cuts
  2. Dissection Cuts
  3. Lifted two-variable cuts
- Dissection cuts are not split cuts
- Lifted two-variable cuts are not split cuts

$conv(X_\alpha)$

1. Recall, $X_\alpha = L_\alpha \cap \mathbb{Z}^2$
2. $conv(X_\alpha) \subseteq \mathbb{R}^2$
3. Extreme points of $conv(X_\alpha)$ are integers
4. How many such polygons exist?

Which one is Cook's example?

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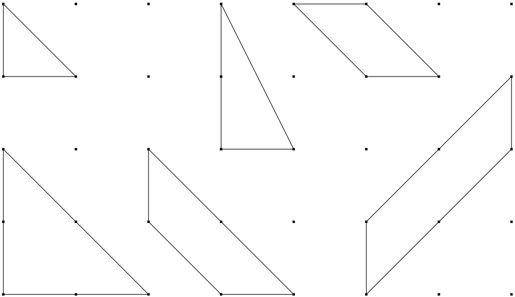

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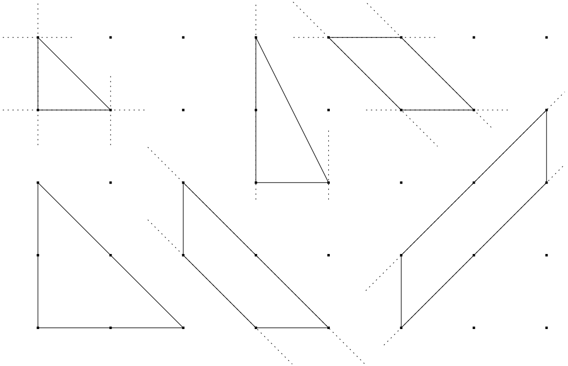

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What about L_α

The main theorem

Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a facet defining inequality that satisfies $\alpha_j > 0$ for all $j \in N$. Then L_α is a polygon with at most four vertices.

Proof: Follows from six lemmas.

Also, there exists a set $S \subseteq N$ such that $|S| \leq 4$ and $\sum_{j \in S} \alpha_j s_j \geq 1$ is facet defining for $conv(P_I(S))$ where,

$$P_I(S) = \{(x, s) \in \mathbb{Z}^n \times \mathbb{R}_+^{|S|} : x = f + \sum_{j \in S} s_j r^j\}$$

Find this inequality and do simultaneous lifting of coefficients for $N \backslash S$ to get the desire cut.

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More notation

- Let $k \leq 4$ denote the number of vertices of $conv(X_\alpha)$.
  $K = \{1, \ldots, k\}$.
- Let the set $\{x^\nu\}_{\nu \in K}$ denote vertices of $conv(X_\alpha)$.
- if $\bar{x} \in X_\alpha$, is not a ray point, then
  $\bar{x} = f + s_{j1} r^{j1} + s_{j2} r^{j2}, s_{j1}, s_{j2} > 0,$ unique
- Such a pair (j_1, j_2) is said to give a representation of $\bar{x}$.
- Additionally if $\alpha_{j1} s_{j1} + \alpha_{j2} s_{j2}$, then (j_1, j_2) is said to give a tight representaion.
- If $cone(\{r^{i1}, r^{i2}\}) \subseteq cone(\{r^{j1}, r^{j2}\})$, then the pair (i_1, i_2) is a sub-cone of (j_1, j_2).
- $T_\alpha(\bar{x}) = \{(j_1, j_2) : (j_1, j_2)$ gives a tight representation of $\bar{x}\}$

Lemma: There exists a unique maximal representation of $(j_1^{\bar{x}}, j_2^{\bar{x}}) \in T_\alpha(\bar{x})$ (One tight representation of $\bar{x}$ can be used).

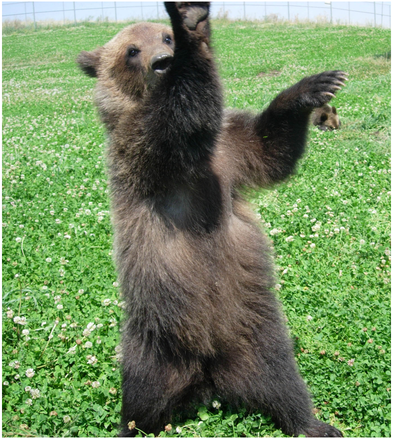
Where does this lead to?

Suppose $\sum_{j \in N} \alpha_j s_j = 1$ is a facet of $conv(P_I)$. Then,

- There exist n affinely independent points in P_I that satisfy this equality,
- Substituting values of s_j and solving for α_j should give this equality as the unique solution
- Project these n points on to plane of (x_1, x_2).
- These projections are either vertices of $conv(X_\alpha)$, or
- they lie on edges of $conv(X_\alpha)$.

So ...

Where does this lead to?



After a lot of hand waving, we get:

- There is a set S, such that $|S| \leq 4$ and
- $\sum_{j \in \alpha} \alpha_j s_j \geq 1$ is facet defining for $P_I(S)$
- $L_\alpha = conv(\{f\} \cup \{\nu^j\}_{j \in S})$

Classification of cuts

- if each vertex of $conv(X_\alpha)$ belongs to a different L_α: *Dissection cut*
- if exactly one facet of L_α contains two vertices of $conv(X_\alpha)$: *Lifted 2-variable* cut
- two facets of L_α contain 2 vertices of $conv(X_\alpha)$ each: split cuts

What kind of cut is Cook's example?