Integer Programming
IE418

Lecture 18

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Reading for This Lecture

- Nemhauser and Wolsey Sections II.1.1-II.1.3, II.1.6
- Wolsey Chapter 8
- Valid Inequalities for Mixed Integer Linear Programs, G. Cornuejols (2006)

Valid Inequalities from Disjunctions

- We continue to focus primarily on the case of a pure integer program

$$\begin{aligned} z_{IP} &= \max\{cx \mid x \in S\}, \\ S &= \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}. \end{aligned}$$

- Valid inequalities for $\mathrm{conv}(S)$ can also be generated based on disjunctions.
- Let $\mathcal{P}_i = \{x \in \mathbb{R}^n_+ \mid A^i x \leq b^i\}$ for $i = 1, \ldots, k$ be such that $S \subseteq \cup_{i=1}^k \mathcal{P}_i$.
- Then inequalities valid for $\cup_{i=1}^k \mathcal{P}_i$ are also valid for $\mathrm{conv}(S)$.

The Union of Polyhedra

- Note that the convex hull of the union of polyhedra is not necessarily a polyhedron.
- Under mild conditions, we can characterize it, however.
- Let Y be the polyhedron described by the following constraints:

$$\begin{aligned} A^i x^i &\leq b^i y_i \;\; \forall i = 1, \ldots, k \\ \sum_{i=1}^{k} x^i &= x \\ \sum_{i=1}^{k} y^i &= 1 \\ y &\geq 0 \end{aligned}$$

- Furthermore, for polyhedron $\mathcal{P}_i$, let $C_i = \{x : A^i x \leq 0$ and let $\mathcal{P}_i = Q_i + C_i$ where Q_i is a polytope.

The Convex Hull of the Union of Polyhedra

- Under the assumptions on the previous slide, we have the following result.

  Proposition 1. *If either* $\cup \mathcal{P}_i = \emptyset$ *or* $C_j \subseteq cone \cup_{i:\mathcal{P}_i \neq \emptyset} C_i$ *for all* j *such that* $\mathcal{P}_j = \emptyset$*, then the following sets are identical:*

  - $\overline{\text{conv}}(\cup_{i=1}^k \mathcal{P}_i\}$
  - $\text{conv}(\cup Q_i) + cone(\cup C_i)$
  - $proj_x Y$.

- Note that the assumptions of the proposition are necessary, but are automatically satisfied if
  - $C^i = \{0\}$ whenever $\mathcal{P}^i = \emptyset$, or
  - all the polyhedra have the same recession cone.

The Convex Hull of the Union of Polyhedra (cont.)

- Note also that if all the polyhedra have the same recession cones, then $\overline{\text{conv}}(\cup_{i=1}^k \mathcal{P}_i) = \text{conv}(\cup_{i=1}^k \mathcal{P}_i)$ and is the projection of

$$
\begin{aligned}
A^i x^i &\leq b^i y_i \ \ \forall i = 1, \ldots, k \\
\sum_{i=1}^k x^i &= x \\
\sum_{i=1}^k y^i &= 1 \\
y &\in \{0, 1\}
\end{aligned}
$$

- This is the case when the polyhedra only differ in their right-hand sides, as is the case when branching on variables.

Valid Inequalities from Disjunctions

Another viewpoint for constructing valid inequalities based on disjunctions comes from the following result:

Proposition 2. *If* $\sum_{j=1}^{n} \pi_j^1 \leq \pi_0^1$ *is valid for* $S_1 \subseteq \mathbb{R}_+^n$ *and* $\sum_{j=1}^{n} \pi_j^2 \leq \pi_0^2$ *is valid for* $S_2 \subseteq \mathbb{R}_+^n$*, then*

$$\sum_{j=1}^{n} \min(\pi_j^1, \pi_j^2)x \leq \max(\pi_0^1, \pi_0^1)$$

for $x \in S_1 \cup S_2$*.*

In fact, all valid inequalities for the union of two polyhedra can be obtained in this way.

Proposition 3. *If* $\mathcal{P}^i = \{x \in \mathbb{R}_+^n \mid A^i x \leq b^i\}$ *for* $i = 1, 2$ *are nonempty polyhedra, then* (π, π_0) *is a valid inequality for* $conv(\mathcal{P}^1 \cup \mathcal{P}^2)$ *if and only if there exist* $u^1, u^2 \in \mathbb{R}^m$ *such* $\pi \leq u^i A^i$ *and* $\pi_0 \geq u^i b^i$ *for* $i = 1, 2$*.*

Strengthening Gomory Cuts Using Disjunction

- Consider again the set of solutions to an IP with one equation.
- This time, we write S equivalently as

$$S = \left\{ x \in \mathbb{Z}_+^n \mid \sum_{j:f_j \le f_0} f_j x_j + \sum_{j:f_j > f_0} (f_j - 1) x_j = f_0 + k \text{ for some integer k} \right\}$$

- Since $k \le -1$ or $k \ge 0$, we have the disjunction

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j - \sum_{j:f_j > f_0} \frac{(1 - f_j)}{f_0} x_j \ge 1$$

OR

$$-\sum_{j:f_j \le f_0} \frac{f_j}{(1 - f_0)} x_j + \sum_{j:f_j > f_0} \frac{(1 - f_j)}{(1 - f_0)} x_j \ge 1$$

The Gomory Mixed Integer Cut

- Applying Proposition 2, we get

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{(1-f_j)}{(1-f_0)} x_j \ge 1$$

- This is called a *Gomory mixed integer* (GMI) cut.
- GMI cuts dominate the associated Gomory cut in general and can also be obtained easily from the tableau.
- In the case of the mixed integer set

$$S = \left\{ x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+ \mid \sum_{j=1}^{p} a_j x_j + \sum_{j=p+1}^{n} g_j x_j = a_0 \right\},$$

the GMI cut is

$$\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{(1-f_j)}{(1-f_0)} x_j + \sum_{j:g_j > 0} \frac{g_j}{f_0} x_j - \sum_{j:g_j < 0} \frac{g_j}{(1-f_0)} x_j \ge 1$$

The GMI closure

- A GMI cut with respect to a polyhedron $\mathcal{P}$ is any cut that can be derived using the above procedure starting from any inequality valid for $\mathcal{P}$.
- The GMI closure is obtained by adding all GMI cuts to the description of $\mathcal{P}$.
- The GMI closure is a polyhedron, but optimizing over it is an $\mathcal{NP}$-hard problem in general.
- It follows that determining whether there is a GMI cut violated by an arbitrary vector is an $\mathcal{NP}$-complete problem.
- Nevertheless, we have just shown that separation of vectors that are basic feasible solutions to a given LP relaxation from the GMI closure can be accomplished in polynomial time.
- The *GMI rank* of both valid inequalities and polyhedra can be defined in a fashion similar to that of the C-G rank (more on this later).

Lift and Project

- Let's now consider $S = \mathcal{P} \cap \mathbb{B}^n$ and assume that the inequalities $x \leq 1$ are included among those in $Ax \leq b$.
- Note that $conv(S) \subseteq conv(\mathcal{P}_j^0 \cup \mathcal{P}_j^1)$ where $\mathcal{P}_j^0 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid x_j = 0\}$ and $\mathcal{P}_j^1 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid x_j = 1\}$ for some $j \in \{1, \ldots, n\}$.
- Applying Proposition 3, we see that the inequality (π, π_0) is valid for $\mathcal{P}_j = conv(\mathcal{P}_j^0 \cup \mathcal{P}_j^1)$ if there exists $u^i \in \mathbb{R}_+^m$, $v^i \in \mathbb{R}_+^n$, and $w^i \in \mathbb{R}_+$ for $i = 0, 1$ such that

$$
\begin{aligned}
\pi &\leq u^0 A + v^0 + w^0 e_j, \\
\pi &\leq u^1 A + v^1 - w^1 e_j, \\
\pi^0 &\geq u^0 b, \\
\pi^0 &\geq u^1 b - w_1,
\end{aligned}
$$

- Notice that this is a set of linear constraints, i.e., we could write a linear program to generate constraints based on this disjunction.

The Cut Generating LP

- This leads to the cut generating LP (CGLP), which generates the most violated inequality valid for $\mathcal{P}_j$.

$$
\begin{aligned}
\text{min} \quad & \pi\hat{x} - \pi^0 \\
\text{subject to} \quad & \pi \le uA + u^0 e_j, \\
& \pi \le vA - v^0 e_j, \\
& \pi^0 \ge ub, \\
& \pi^0 \ge vb - v_0, \\
& \sum_{i=1}^{m} u_i + u_0 + \sum_{i=1}^{m} v_i + v_0 = 1 \\
& u, u_0, v, v_0 \ge 0
\end{aligned}
$$

- The last constraint is just for normalization.
- This shows that the separation problem for $\mathcal{P}_j$ is polynomially solvable.

Gomory Cuts vs. Lift-and-Project Cuts

- Note that all Gomory cuts are lift-and-project cuts.
- In fact, there is a direct correspondence between basic feasible solutions of the CGLP and basic (possibly infeasible) solutions of the usual LP relaxation.
- By pivoting in the LP relaxation, we can implicitly solve the cut generating LP (see Balas and Perregaard).
- Thus, the procedure for generating lift-and-project cuts is almost exactly the same as that for generating Gomory cuts.

Another Derivation

- Consider the following procedure:
  1: Select $j \in \{1, \ldots, n\}$.
  2: Generate the nonlinear system $x_j(Ax-b) \geq 0$, $(1-x_j)(Ax-b) \geq 0$.
  3: Linearize the system by substituting y_i for $x_i x_j, i \neq j$, and x_j for x_j^2. Call this polyhedron M_j.
  4: Project M_j onto the x-space.
- In this case, the resulting polyhedron is again $\mathcal{P}_j$.
- This procedure can be strengthened in a number of different ways.

The Lift-and-Project Closure

- The lift-and-project closure is

$$\mathcal{P}^1 = \cap_{j=1}^{n} \mathcal{P}_j$$

- We have just shown that optimization over the lift-and-project closure can be accomplished in polynomial time.
- Let $\mathcal{P}^k$ be the lift-and-project closure of $\mathcal{P}^{k-1}$ for $k > 1$.
- The lift-and-project rank of $\mathcal{P}$ is the smallest number k such that $\mathcal{P}^k = \text{conv}(S)$.
- Surprisingly, the lift-and-project rank is bounded by n.
- Note that these results apply only to binary and mixed binary integer programs.

Split Inequalities

- Let $\pi \in \mathbb{Z}^n_+$ and $\pi_0 \in \mathbb{Z}$ be given and define

$$\begin{aligned} \mathcal{P}^1 &= \mathcal{P} \cap \{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\} \\ \mathcal{P}^2 &= \mathcal{P} \cap \{x \in \mathbb{R}^n \mid \pi x \geq \pi_0 + 1\} \end{aligned}$$

- Any inequality valid for $\text{conv}(\mathcal{P}_1 \cup \mathcal{P}_2)$ is valid for S and is called a *split cut*.
- The *split closure* is the set of points satisfying all possible split cuts and is a polyhedron.
- In fact, the split closure and the GMI closure discussed earlier are identical.
- We can define the *split rank* of an inequality and of a polyhedron as before.
- In the pure integer case, the split rank (and GMI rank) of $\mathcal{P}$ is finite, but it may not be in the mixed case.
- In the mixed binary case, the split rank is bounded by n.

Valid Inequalities for Mixed-Integer Sets

- So far, we have been dealing almost exclusively with polyhedra in which all variables have to be integer.
- We want to develop a procedure analogous to C-G for mixed-integer sets.

Proposition 4. *Let* $T = \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} \mid \sum_{j=1}^p a_j x_j + \sum_{j=p+1}^n g_j y_j \leq b\}$*, where* $a_j \in \mathbb{Q}$ *for* $0 \leq j \leq p$*,* $g_j \in \mathbb{Q}$ *for* $p+1 \leq j \leq n$*, and* $b \in \mathbb{Q}$*. Then the inequality*

$$\sum_{j=1}^p \lfloor a_j \rfloor x_j + \frac{1}{1-f_0} \sum_{j: g_j < 0} g_j y_j \leq \lfloor b \rfloor.$$

- In fact, if $a_j \in \mathbb{Z}$, $gcd\{a_1, \ldots, a_n\} = 1$, and $b \notin \mathbb{Z}$, then the above inequality is facet-inducing for T.

Mixed-Integer Rounding Procedure

- Now consider the general mixed-integer set

$$T = \{x \in \mathbb{Z}^n_+, y \in \mathbb{R}^p_+ \mid Ax + Gy \leq b\}$$

- Given two valid inequalities

$$\sum_{j \in N} \pi^i_j x_j + \sum_{j \in J} \mu^i_j y_j \leq \pi^i_0 \text{ for } i = 1, 2,$$

we can construct a third inequality

$$\sum_{j \in N} \lfloor \pi^2_j - \pi^1_j \rfloor x_j + \frac{1}{1 - f_0} \left(\sum_{j \in N} \pi^1_j x_j + \sum_{j \in J} \min(\mu^1_j, \mu^2_j) y_j - \pi^1_0 \right) \leq \lfloor \pi^2_0 - \pi^1_0 \rfloor,$$

where $\pi^2_0 - \pi^1_0 = \lfloor \pi^2_0 - \pi^1_0 \rfloor + f_0$.