L-shaped Decomposition of 2-stage SPs with Integer Recourse

Cor@l Seminar Series

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References

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Outline

- Problem Introduction
- Generalized duality
- Generalize L-shaped decomposition
- Dual function generation algorithms
- Future work

2-stage Stochastic Programs with Integer Recourse

Consider the following stochastic problem

$$\min_{x \in X} cx + \mathbb{E}_\xi \min \left\{ qy | T(\xi)x + Wy \geq h(\xi), y \in \mathbb{Z}_+^{n_2} \right\} \tag{1}$$

where ξ is a random variable having support $\Xi \subset \mathbb{R}^k$ and

$$X = \{x \in \mathbb{R}_+^{n_1} | Ax \geq b\}.$$

Comment 1. *The part of the objective function and the constraints only related to the first stage decision variable x form a LP. This is only for simplicity.*

Deterministic Equivalent

We make the following assumption

- The random variable ξ has a discrete distribution with finite support, say $\Xi = \left\{\xi^1, \ldots, \xi^r\right\}$ and $P\left(\xi = \xi^j\right) = p^j$.

Under this assumption, (1) is equivalent to

$$\begin{aligned} \min \quad & cx + \sum_{j=1}^{r} p^j q y^j \\ \text{s.t.} \quad & Ax \geq b \\ & T(\xi)x + Wy \geq h(\xi), \quad j = 1, \ldots, r \\ & x \in \mathbb{R}_+^{n_1}, y \in \mathbb{Z}_+^{n_2} \end{aligned} \tag{2}$$

where the constraints have a dual blockangular structure or L-shaped form.

Comment 2. *(2) has $n_1 + rn_2$ variables.*

Reformulation

Rewriting the problem in terms of only first stage variables yields:

$$\min\{cx + \mathcal{Q}(x) | x \in X\} \tag{3}$$

where

$$\mathcal{Q}(x) := \mathbb{E}_\xi \Phi(h(\xi) - T(\xi)x) = \sum_{j=1}^{r} p^j \Phi(h(\xi) - T(\xi)x)$$

and Φ is the *value function* of the second stage problem

$$\Phi(d) = \min\{qy | Wy \geq d, y \in \mathbb{Z}_+^{n_2}\}, d \in \mathbb{R}^{m_2}. \tag{4}$$

Comment 3. $\mathcal{Q}(x)$ *is nonconvex and discontinuous.*

Definition Review

- A function F is said to be *nondecreasing* if

$$F(a) \leq F(b) \quad \forall a, b \in \mathbb{R}^m, a \leq b$$

- A function F is said to be *superadditive* if

$$F(a) + F(b) \leq F(a + b), \quad \forall a, b \in \mathbb{R}^m$$

- The *round-up* $\lceil F \rceil$ is defined by $\lceil F \rceil(d) = \lceil F(d) \rceil$, where $\lceil F(d) \rceil$ is smallest integer larger than $F(d)$.

Generalized Dual

- Recall the second stage problem

$$\Phi(d) = \min\{qy|Wy \geq d, y \in \mathbb{Z}_+^{n_2}\}, d \in \mathbb{R}^{m_2}.$$

- Let $\overline{\mathcal{F}}$ be the set of all functions $F : \mathbb{R}^{m_2} \rightarrow \overline{\mathbb{R}}$ that satisfy $F(0) = 0$ and are nondecreasing.
- Then, we can define the dual of the problem as

$$\begin{aligned} \max_F \quad & F(d) \\ \text{s.t.} \quad & F(Wy) \leq qy, \quad \forall y \in \mathbb{Z}_+^{n_2} & (5) \\ & F \in \mathcal{F} & (6) \end{aligned}$$

where $\mathcal{F}$ is a subset of $\overline{\mathcal{F}}$.

IP Duality and Farkas' Lemma

Theorem 1. [Weak Duality] *$qy \leq F(d)$ for all feasible solutions y of (4) and all dual feasible functions F of (5).*

Theorem 2. *If the function class $\mathcal{F}$ is suitably large then (4) is infeasible if and only if $\exists \hat{G} \in \mathcal{F}$ with $\hat{G}(Wy) \leq 0$ for all $y \in \mathbb{Z}_+^{n_2}$ and $\hat{G}(d) > 0$. The function $\hat{G}$ is then called a dual ray. If (4) is feasible, then $\hat{y}$ is optimal in (4) if and only if $\exists \hat{F} \in \mathcal{F}$ feasible in (5) such that $q\hat{y} = \hat{F}(d)$.*

- This result is analogous to the Strong Duality Theorem and Farkas' Lemma for linear programming.

Generalized L-shaped Decomposition

- We rewrite (3) as

$$\min\{cx + \theta | \theta \geq \mathcal{Q}(x), x \in X\} \tag{7}$$

and represent the constraint $\theta \geq \mathcal{Q}(x)$ by means of dual price functions.

- For each outcome $\xi^j \in \Xi$, we have a second stage problem

$$\min\{qy | Wy \geq h(\xi^j) - T(\xi^j)x, y \in \mathbb{Z}_+^{n_2}\} \tag{8}$$

and associated dual

$$\max_F\{F(h(\xi^j) - T(\xi^j)x) | F(Wy) \leq qy, \quad \forall y \in \mathbb{Z}_+^{n_2}, F \in \mathcal{F}\}. \tag{9}$$

Feasiblity and Optimality Cuts

- We generate feasibility cuts of the form

$$\hat{G}(h(\xi^j) - T(\xi^j)x) \leq 0$$

where $\hat{G}$ is the optimal dual solution of the Phase I problem:

$$\min\{et|Wy + It \geq h(\xi^j) - T(\xi)x^*\}$$

- By solving (9) with $x = x^*$ for each $\xi^j \in \Xi$, we generate optimality cuts of the form

$$\theta \geq \sum_{j=1}^{r} p^j \hat{F}^j (h(\xi^j) - T(\xi^j)x)$$

where $\hat{F}^j, j = 1, \ldots, r$ are optimal solutions of (9).

Relaxed Master Problem

At each iteration, we solve the *current problem*:

$$
\begin{aligned}
\min \quad & cx + \theta \\
\text{s.t} \quad & 0 \geq \hat{G}_{k_j}(h(\xi^j) - T(\xi^j)x), \quad k_j = 1, \ldots, s(j), j = 1, \ldots, r \\
& \theta \geq \sum_{j=1}^{r} p^j \hat{F}^j(h(\xi^j) - T(\xi^j)x) \quad k = 1, \ldots, t \\
& x \in X
\end{aligned} \tag{10}
$$

We denote solutions to 10 by (x^n, θ^n). The algorithm terminates when $cx^n + \theta^n = \overline{z}^n$, or (10) is infeasible.

Comment 4. *(10) has $n_1 + 1$ variables, but a lot of constraints.*

Cutting Plane Algorithm

Let $\mathcal{F}$ be the set of nondecreasing superadditive functions such that $F(0) = 0$. Then, (9) is equivalent to

$$\begin{aligned} \max \quad & F(h(\xi^j) - T(\xi^j)x) \\ \text{s.t.} \quad & F(w_j) \le q_j, \quad j = 1, \ldots, n_2 \\ & F \in \mathcal{F} \end{aligned} \tag{11}$$

In a cutting plane procedure

- Valid inequalities are successively generated and added to the constraint set
- LP relaxation are solved
- Process is repeated until current LP-solution is integral
- Cuts are of the form

$$\sum_{j=1}^{n_2} F^{(l)}(w_j) y_j \ge F^{(l)}(q), \; l = 1, \ldots, \tau$$

Cutting Plane Algorithm (2)

At termination, we have the function

$$F(d) := \sum_{i=1}^{m_2} u_i d_i + \sum_{i=1}^{\tau} u_{m_2+i} F^{(i)}(d)$$

that is a feasible and optimal solution of (11), where dual variables

$$(u_1, \ldots, u_{m_2}, u_{m_2+1}, \ldots, u_{m_2+r})$$

are obtained from the LP-solution.

Branch and Bound

Alternatively, we can solve the second stage problems using a branch-and-bound algorithm. In this case, we generate price functions of the form

$$F(d) := \min_{i=1,\ldots,P} \{u^i d + b^i\},\ \ u^i = (u^i_1, \ldots, u^i_{m_2}) \geq 0$$

for some finite $P \in \mathbb{N}$.

We generate these functions by solving the dual of

$$\min\{qy \mid Wy \geq d, k^i \leq y \leq l^i\},$$

for terminal node i and RHS d, given by

$$\max\{ud + \underline{u}k^i - \overline{u}l^i \mid uW + \underline{u} - \overline{u} \leq q, u, \underline{u}, \overline{u} \geq 0\}$$

and letting $f_i(d) = u^i d + \underline{u}^i k^i - \overline{u}^i l^i = u^i d + b^i$.

Stay Tuned...

- Apply a similar idea to the MIP Interdiction Problem (MIPINT):

$$\begin{aligned} \min_{x \in X} \quad & cx + dy + \max_{y} hy \\ \text{subject to} \quad & Ey \leq g \\ & y \leq u(1-x) \\ & y \in Y_{INT} \end{aligned} \tag{12}$$

  where $X = \{x : Ax \geq b, x \in \mathbb{B}^n\}$ and $Y_{INT} \subseteq \mathbb{R}^n_+$ defines some integrality conditions on the lower-level variables.

- Using inner approximation, rather than outer approximation for lower-level problem
  - Maybe a Dantzig-Wolfe-like scheme