Solving Symmetric Integer Programs

Jim Ostrowski

ISE Department
COR@L Lab
Lehigh University
jao204@lehigh.edu

COR@L
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

Some Preliminaries

- For a set $S \subseteq I^n$, the orbit of S with respect to Γ is the set of all subsets of I^n to which S can be sent by permutations in Γ:

$$\mathrm{orb}(S, \Gamma) \stackrel{\mathrm{def}}{=} \{S' \subseteq I^n \mid \exists \pi \in \Gamma \text{ such that } S' = \pi(S)\}.$$

- I care (mostly) about orbits of sets of cardinality one, corresponding to decision variables x_j
- By definition, if $j \in \mathrm{orb}(\{k\}, \Gamma)$, then $k \in \mathrm{orb}(\{j\}, \Gamma)$, i.e. the variable x_j and x_k share the same orbit. Therefore, the union of the orbits

$$\mathcal{O}(\Gamma) \stackrel{\mathrm{def}}{=} \bigcup_{j=1}^{n} \mathrm{orb}(\{j\}, \Gamma)$$

  forms a partition of $I^n = \{1, 2, \ldots, n\}$, which we refer to as the orbits of Γ.
- The orbits encode which variables are "equivalent" with respect to the symmetry Γ.

Constraint Orbital Branching—The Whole Idea

- Let $O \in \mathcal{C}(A)$ be an orbit of the symmetry group of A representing constraints, h any element in O.
- Surely we can branch on the disjunction $c_h x = b \ \vee \ \{c_j x \geq b + 1 | \forall j \in O\}$

COR@L COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

Basic Idea

- So now I have subproblems with equalities in them...
- Use them!
- Create a relaxation to the subproblem by removing all variables not included in a chosen equality constraint and remove all constraints which include a removed variable.
- Find the collection of all non-isomorphic solutions to the relaxation.
- Use these solutions as partial solutions to the original subproblem, branch on these solutions.

Steiner Triple Systems

- Let X be a set of $v \geq 3$ elements.
- B is collection of 3 elements subsets of X s.t. every pair of elements of X is found in exactly one element of B.
- $S_3 = \{\{1, 2, 3\}\}$
- $S_7 = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$
- A Steiner Triple System exists if $v = 1,\ 3 \bmod (6)$.

Growing Triple Systems

- Lets build S_9 using $S_3 = \{\{1,\ 2,\ 3\}\}$.

Growing Triple Systems

- Lets build S_9 using $S_3 = \{\{1, 2, 3\}\}$.
- First, start with 3 S_3's, $X_1 = \{1, 2, 3\}$, $X_2 = \{4, 5, 6\}$, $X_3 = \{7, 8, 9\}$ and their corresponding triple $\{\{1, 2, 3\}\}, \{\{4, 5, 6\}\}, \{\{7, 8, 9\}\}$
- Link "like elements" with sets $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}$ (i.e. form a set with all the first elements of each X, second elements, ...).
- Link remaining elements using solution to S_3. Using $\{\{1, 2, 3\}\}$, create sets by choosing the first element from set X_i, the second element from set X_j $(j \neq i)$, and the third element from set X_k $(k \neq i, j)$.

COR@L
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

Growing Triple Systems

$$A_9 = \begin{bmatrix} 1_{s_1} & 2_{s_1} & 3_{s_1} & 1_{s_2} & 2_{s_2} & 3_{s_2} & 1_{s_3} & 2_{s_3} & 3_{s_3} \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

A Hard Integer Program

- $$\min \sum_{i=1}^{7} x_i$$
  $$\text{s.t.} \quad x_1 + x_2 + x_4 \geq 1$$
  $$x_2 + x_3 + x_5 \geq 1$$
  $$x_3 + x_4 + x_6 \geq 1$$
  $$x_4 + x_5 + x_7 \geq 1$$
  $$x_1 + x_5 + x_6 \geq 1$$
  $$x_2 + x_6 + x_7 \geq 1$$
  $$x_7 + x_1 + x_3 \geq 1$$
  $$x \in \{0, 1\}^n$$

COR@L
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

A Hard Integer Program

- For sts7, for every triple $\{a, b, c\}$ in S_7, create a constraint

$$x_a + x_b + x_c \geq 1$$

- These problems are very difficult.
- The smallest unsolved instance has only 135 variables
- Why are these problems so difficult?
- These problems have a large degree of symmetry, but that is not all.
- The LP relaxation of sts135 is 45, the smallest known feasible solution has value 103.
- A large gap between the relaxation and the optimal solution make integer programs very hard to solve.

Improving the Gap

- sts135 was created by using 3 sts45 problems (implying there is symmetry present in the problem)
- The optimal solution of sts45 is 30, so...

Improving the Gap

- sts135 was created by using 3 sts45 problems (implying there is symmetry present in the problem)
- The optimal solution of sts45 is 30, so...
- We kow that $\sum_{i=1}^{45} x_i \geq 30$, $\sum_{i=46}^{90} x_i \geq 30$, and $\sum_{i=91}^{135} x_i \geq 30$
- Adding these inequalities increases the LP relaxtion to 90.
- Nice, but we can do better!

Exploiting Symmetry of Constriants

- Similar to variables, constraints can be symmetric.
- We can use orbital branching to "branch" on the constraints.

Exploiting Symmetry of Constriants

- Similar to variables, constraints can be symmetric.
- We can use orbital branching to "branch" on the constraints.
- So... Either $\sum_{i=1}^{45} x_i = 30$ or $\sum_{i=1}^{45} x_i \geq 31$, $\sum_{i=46}^{90} x_i \geq 31$, and $\sum_{i=91}^{135} x_i \geq 31$
- So what good does this do?

But Wait, There is More!

- How many sts45's are in an sts135?
- The constraint $\sum_{i=1}^{45} x_i \geq 30$ is also equivalent to the constraints:

$$\sum_{i=1}^{15} x_i + \sum_{i=46}^{60} x_i + \sum_{i=91}^{105} x_i \geq 30$$

$$\sum_{i=1}^{15} x_i + \sum_{i=61}^{75} x_i + \sum_{i=121}^{135} x_i \geq 30$$

$$\sum_{i=1}^{15} x_i + \sum_{i=76}^{90} x_i + \sum_{i=106}^{120} x_i \geq 30$$

$$\sum_{i=16}^{30} x_i + \sum_{i=46}^{60} x_i + \sum_{i=121}^{135} x_i \geq 30$$

$$\sum_{i=16}^{30} x_i + \sum_{i=61}^{75} x_i + \sum_{i=106}^{120} x_i \geq 30$$

$$\sum_{i=16}^{30} x_i + \sum_{i=76}^{90} x_i + \sum_{i=91}^{105} x_i \geq 30$$

45 60 120

Exploiting Symmetry of Constriants

- Using orbital branching we can generate all non-isomorphic solutions to sts45 with value 30
- (There is only 1 of them)
- The subproblem formed by setting $\sum_{i=1}^{45} x_i = 30$ can solved by fixing the first 45 variables to correspond to the solution of sts45.
- Lather, rinse, repeat...
- Keep branching on constraints until we increase rhs to 34
- Why? With the rhs of 35, the LP relaxation is 105, so these problems cannot contain a solution of size 103 or better

How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30

How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
- 246 solutions of sts45 of value 31

COR@L
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
- 246 solutions of sts45 of value 31
- 9497 solutions of sts45 of value 32

How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
- 246 solutions of sts45 of value 31
- 9497 solutions of sts45 of value 32
- 61539 solutions of sts45 of value 33

How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
- 246 solutions of sts45 of value 31
- 9497 solutions of sts45 of value 32
- 61539 solutions of sts45 of value 33
- 122972 solutions of sts45 of value 34
- ... CRAP! ...

But are all of them non-isomorphic?

- NO! By checking for isomorphism we can remove...

But are all of them non-isomorphic?

- NO! By checking for isomorphism we can remove...
- 13 of the solutions for sts45 of values 30-32...
- Yippee, I saved 2 hours of computation time!!!

But are all of them non-isomorphic?

- NO! By checking for isomorphism we can remove...
- 13 of the solutions for sts45 of values 30-32...
- Yippee, I saved 2 hours of computation time!!!
- I can do much, much better!

Exploiting Symmetry of Constriants

- Pros: Each subproblem will have 45 variables fixed and are much easier to solve.
- This can be done easily in parallel, different computers can solve different subproblems independently of each other.
- Cons: In order to solve sts135 we need to generate all non-isomorphic solutions to sts45 with values 30-34
- There can be a whole lot of these.
- Generating these may take awhile.