Mixing mixed-integer inequalities

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C@RAL Seminar Series
April, 2007

References

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- M. Zhao and I. Farias, The polar of a simple mixed-integer set.

Introduction

Mixed-integer programming problem:

$$z = \min\{cx \ : \ x \in S\}$$

where

$$S = \{x \in \mathcal{R}^{m_1} \times \mathcal{Z}^{m_2} : Ax \leq b\}$$

How to obtain strong valid inequalities for S ?

- General purpose cutting planes, special purpose cutting planes based on structure, relaxations.
- MIR inequalities...
- Obtain new classes of valid inequalities by using known classes
- lifting, mixing.
- Mixing: generating new valid inequalities by combining known MIR inequalities.

Base inequalities

Given a mixed integer region $S \subseteq \mathcal{R}^{m_1} \times \mathcal{Z}^{m_2}$ and a collection of $m \geq 2$ valid inequalities for S:

$$f^i(x) + Bg^i(x) \geq \pi^i \quad i \in \mathcal{I} = \{1, \ldots, m\} \tag{1}$$

where

$$B \in \mathcal{R}^1_+, \pi \in \mathcal{R}^1, f^i(x) \geq 0, \text{ and } g^i(x) \in \mathcal{Z}$$

- valid inequalities of this form are called *base inequalities* from now on.

MIR inequality for the base inequality

- For $i \in \mathcal{I}$, let $\tau^i = \lceil \pi^i / B \rceil$, and $\gamma^i = \pi^i - (\tau^i - 1)B$.
- $\tau^i \in \mathcal{Z}^1$ and $B \geq \gamma^i > 0$.

Theorem

For any $i \in \mathcal{I}$, the so-called simple MIR inequality

$$f^i(x) \geq \gamma^i(\tau^i - g^i(x)) \tag{2}$$

is valid for S.

Proof:

$$f^i(x) \geq \pi^i - Bg^i(x) = \gamma^i + B(\tau^i - g^i(x) - 1).$$

- For $x \in S$, either $g^i(x) \geq \tau^i$ or $g^i(x) \leq \tau^i - 1 \ldots$ □

MIR inequality...

- Nemhauser and Wolsey method for generating MIR on the set H:

$$
\begin{aligned}
&H = \{(f,g) \in \mathcal{R} \times \mathcal{Z} : -f \leq 0, -\frac{f}{B} - g \leq -\frac{\pi}{B}\} \\
&\Rightarrow -g - \frac{f}{\pi - B(\lceil \pi/B \rceil - 1)} \leq \lceil \pi/B \rceil, \\
&\equiv f \geq (\pi - B(\lceil \pi/B \rceil - 1))(\lceil \pi/B \rceil - g).
\end{aligned}
$$

- $\Rightarrow$ (2) is an MIR inequality.
- This inequality suffices to obtain the complete linear description of $conv(H)$.
- Now try this on mixed-integer sets with more than 1 base inequality.

Mixing procedure

- Let $I = \{1, \ldots, n\} \subseteq \mathcal{I} = \{1, \ldots, m\}$ be a subset of base inequalities.
- wlog, $\gamma^i \geq \gamma^{i-1}$ for all $n \geq i \geq 2$.
- given $\bar{f}(x) \in R^1 : \bar{f}(x) \geq f^i(x) \geq 0 \; \forall x \in S, \; \forall i \in I$,

Theorem

The following mixed MIR inequalities

$$\bar{f}(x) \geq \sum_{i=1}^{n} (\gamma^i - \gamma^{i-1})(\tau^i - g^i(x)) \tag{3}$$

and

$$\bar{f}(x) \geq \sum_{i=1}^{n} (\gamma^i - \gamma^{i-1})(\tau^i - g^i(x)) + (B - \gamma^n)(\tau^1 - g^1(x) - 1) \tag{4}$$

where $\gamma^0 = 0$, *are valid for* S.

Mixing procedure...

Proof:

- For any fixed $\bar{x} \in S$, define $\beta = \max_{i \in I}\{\tau^i - g^i(\bar{x})\}$ and
- $v = max\{i \in I : \beta = \tau^i - g^i(\bar{x})\}$.
- If $\beta \leq 0$, RHS of (3)-(4) is at most 0.
- Therefore assume that $\beta = \tau^v - g^v(\bar{x}) \geq 1$.
- Using $\beta \geq \tau^i - g^i(\bar{x})\ \forall i \leq v$ and $\beta \geq \tau^i - g^i(\bar{x}) + 1\ \forall i > v$,
- $\ldots$ □

When $|I| = 1$ and $\bar{f} = f^1$, then (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1).

Example

- Let $S = \{(x, y) \in \mathcal{R}^1_+ \times \mathcal{Z}^2 : x_1 + 10y_1 \geq 3, x_1 + 10y_2 \geq 5\}$
- MIR inequalities associated with the base inequalities:

$$x_1 \geq 3(1 - y_1)$$
$$x_1 \geq 5(1 - y_2)$$

- Mixed MIR inequalities with $\bar{f} = f^1 = f^2 = x_1$ and $g^i = y_i$:

$$x_1 \geq 3(1 - y_1) + 2(1 - y_2)$$
$$x_1 \geq 5(1 - y_2) + 2(1 - y_2) + 5(-y_1)$$

- All these inequalities together yield $conv(S)$.
- Strength of (3)-(4) depends on choice of $\bar{f}$.
- $\bar{f}(x) \geq f^*(x) = \max_{i \in I} f^i(x)$
- $f^*(x)$ not smooth in general.

Generating base inequalities - example

Fixed charge capacitated network problem.

- Aggregate flow balance constraints over a set of connected nodes...
- $\sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j = d$,
- $\Rightarrow \sum_{j \in N_1} x_j \geq d$
- Partition of $N_1 = \{F, G\}$, using variable upper bound constraints $x_j \leq u_j x_j$ for $j \in G$ gives
- $\sum_{j \in F} x_j + \sum_{j \in G} u_j y_j \geq d$.
- Similar approach can be followed for other problems.

General purpose mixing from the simplex tableau

Solve the linear programming relaxation and rewrite the problem using the optimal basis:

$$z = \min\{cx : x_i + \sum_{j \in NV} \bar{a}_{ij} x_j = \bar{b}_i, x \geq 0, x \in \mathcal{Z}^n\}.$$

- Define several base inequalities for each row.
- for a fixed $i \in BV$ and coefficient $\sigma \in \mathcal{Z}$, define

$$g(x) = \sigma x_i + \sum_{j \in NV} \lfloor \sigma \bar{a}_{ij} \rfloor x_j$$

$$f(x) = \sum_{j \in NV} (\sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor) x_j$$

- Relax to $f(x) + g(x) \geq \sigma \bar{b}_i$ ($B = 1$).
- When $\sigma = 1$, (2) $\equiv$ GMI cut.

Strengthening MIR and Mixed MIR inequalities

For a fixed $i \in BV$ and $\sigma \in cZ$, let $\tau = \lceil \sigma \bar{b}_i \rceil$ and $\gamma = \sigma \bar{b}_i - (\tau - 1)$

- Associated MIR inequality:

$$\sigma \gamma x_i + \sum_{j \in NV} [(\sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor) x_j + \gamma \lfloor \sigma \bar{a}_{ij} \rfloor] x_j \geq \gamma \tau \qquad (5)$$

- if $(\sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor) > \gamma$ for some $j \in NV$, then redefine $g(x)$ and $f(x)$:

$$g(x) = \sigma x_i + \sum_{j \in NV : \sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor > \gamma} \lceil \sigma \bar{a}_{ij} \rceil x_j + \sum_{j \in NV : \sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor \leq \gamma} \lfloor \sigma \bar{a}_{ij} \rfloor x_j$$

Strengthening MIR and Mixed MIR inequalities...

$$f(x) = \sum_{j \in NV : \sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor \leq \gamma} (\sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor) x_j$$

This gives the following MIR inequality (2):

$$\sigma \gamma x_i + \sum_{j \in NV} [\min(\sigma \bar{a}_{ij} - \lfloor \sigma \bar{a}_{ij} \rfloor, \gamma) + \gamma \lfloor \sigma \bar{a}_{ij} \rfloor] x_j \geq \gamma \tau$$

- This is clearly stronger than (5).
- For mixed MIR inequalities: same approach but more involved.
- pick a threshold value $\beta_j \in \mathcal{R}$ for each $j \in NV$ and relax the base inequality $f(x) + g(x) \geq \sigma \bar{b}_i$ if coefficient of x_j in $f(x)$ is greater than β_j...

Strength of Mixing procedure

Possible to generalize (3) and (4)...

Lemma

The following inequality is valid for S

$$\bar{f}(x) + \alpha \geq \sum_{i \in \mathcal{I}} \delta^i (\tau^i - g^i(x)) \qquad (6)$$

provided:

1. $(\delta, \alpha) \in \mathcal{P}^C$, *where*

$$\mathcal{P}^C = \{(\delta, \alpha) \in \mathcal{R}_+^{|\mathcal{I}|+1} : \sum_{i \in \mathcal{I}} \delta^i \leq B$$
$$\sum_{j \leq i} \delta^j \leq \alpha + \gamma^i, \ \forall i \in \mathcal{I}\}$$

2. $\bar{f}(x) \geq f^i(x)$ *for all* $x \in S, i \in \mathcal{I}$ *with* $\delta^i > 0$.

Strength of Mixing procedure...

(3) and (4) contain all important inequalities of form (6) with $(\delta, \alpha) \in \mathcal{P}^C$.

Lemma

Let $p = (\delta, \alpha)$ *be an arbitrary non-zero extreme point of* $\mathcal{P}^C$, *define* $I = \{i \in \mathcal{I} : \delta^i > 0\} = \{i_1, i_2, \ldots, i_n\}$ *with* $i_1 < i_2 < \ldots < i_n$. *The extreme point* $p = (\delta, \alpha)$ *is characterized by:*

$$\alpha \in \{0, B - \gamma^{i_n}\}$$
$$\delta^{i_1} = \gamma^{i_1} + \alpha$$
$$\delta^{i_j} = \gamma^{i_j} - \gamma^{i_{j-1}} \; j = 2, \ldots, n$$
$$\delta^i = 0 \text{ for } i \in \mathcal{I} \backslash I.$$

(6) generated by $(\delta, \alpha) \in \mathcal{P}^C$ is equivalent to or dominated by positive combinations of (3) and (4).

Finding violated inequalities

- If $\bar{f}$ is fixed or known apriori, given a point $\bar{x} \in \mathcal{R}^{m_1} \times \mathcal{Z}^{m_2}$,
- separation problem is finding $(\hat{\delta}, \hat{\alpha}) \in \mathcal{P}^C$ that maximizes the RHS of

$$\bar{f}(\bar{x}) \geq \sum_{i \in \mathcal{I}} \hat{\delta}^i(\tau^i - g^i(\bar{x})) - \hat{\alpha}. \tag{7}$$

- Let $h^i(x) = \tau^i - g^i(x)$. Let $(\hat{\delta}, \hat{\alpha})$ be such that it has minimum number of non-zero components.
- Define $\hat{I} = \{i \in \mathcal{I} : \hat{\delta}^i > 0\} = \{i_1, \ldots, i_n\}$ with $i_1 < i_2 < \ldots < i_n$.
- $\hat{\alpha} \neq 0 \Leftrightarrow \max_{j \in \mathcal{I}}\{h^j(x)\} > 1$.

Finding violated inequalities...

- For all $i \in \mathcal{I}$, $i \in \hat{I} \Leftrightarrow$

$$h^i(\bar{x}) > h^j(\bar{x}) \; \forall j > i$$
$$h^i(\bar{x}) > \max_{j \in \mathcal{I}} [h^j(\bar{x}) - 1]$$
$$h^i(\bar{x}) > 0$$

- The optimal $\hat{I}$ must satisfy

$$h^{i_1}(\bar{x}) > h^{i_2}(\bar{x}) > \ldots > h^{i_n}(\bar{x}) > \max\{h^{i_1}(\bar{x}) - 1, 0\}$$

- Can be found easily by computing and sorting $h^i(\bar{x})$.
- Values $\hat{\delta}^i$ for $i \in \hat{I}$ can be fixed using previous lemma.
- If $n \geq 1$ and $h^{i_1}(\bar{x}) > 1$, fix $\hat{\alpha} = B - \max_{j \in \hat{I}}\{\gamma^j\}$ and 0 otherwise.

Variations

- When $f^i(x) < 0$ for base inequality $i \in \mathcal{I}$, MIR (2) not valid.
- If we know a lower bound, $f^i(x) \geq LB^i$, then we can rewrite the base inequality:

$$(f^i(x) - LB^i) + Bg^i(x) \geq \pi^i - LB^i$$

- If base inequalities have the form

$$f^i(x) + B^i g^i(x) \geq \pi^i$$

  then we define $\tau^i = \lceil \pi^i / B^i \rceil$ and $\gamma^i = \pi^i - B^i(\tau^i - 1)$.
- check if $\hat{B} = \min_{i \in I}\{B^i\} \geq \bar{\gamma} = \max_{i \in I}\{\gamma^i\}$
- relax base inequalities with small B^i's by replacing with $\hat{\gamma}$.
- Another approach is to scale inequalities to increase $\bar{B}$ or reduce $\bar{\gamma}$.

Mixing independent constraints

Next time... along with some more stuff... :)