Pancakes: Half-baked and burnt

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COR@L Seminar Series, Spring 2005

Outline

- William H. Gates, C.H. Papadimitriou, *Bounds for sorting by prefix reversal*, Discrete Mathematics, 27, 1979
- M.H. Heydari, I.H. Sudborogh, *On the diameter of the pancake network*, Journal of Algorithms, 25, 1997

The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them so that the smallest winds up on top, and so on, down to the largest at the bottom, by grabbing several from the top and flipping them over, repeating this varying the number I flip, as many times as necessary. If there are n pancakes, what is the maximum number of flips as a function of n, that I will ever have to use to rearrange them?

Trivial Bounds

- **lower bound:** for $n \geq 4$, $f(n) \geq n$
- M.R. Garey et. al.showed that $f(n) \geq n + 1$ for $n \geq 6$.
- **upper bound:** $f(n) \leq 2n$.

- They showed that $f(n) \leq (5n+5)/3$ by designing an algorithm.

Some Notation

- **flip:** taking some pancakes from the top and flipping them over.
- **adjacency:** a pair of pancakes that are adjacent in the stack,and no other pancake has size intermediate between the two. (It is assumed that the first and the last pancakes are adjacent)
- **block:** adjacency closure
- **free:** if a pancake is not in a block, it is free.
- S_n**:** Set of possible permutations for a pancake problem of size n.

The Algorithm

input: a permutation $\pi \in S_n$. **output:** a permutation σ with n-1 adjacencies. **o:** stands for one of {1,-1} Repeat the Following:

- **1** t is free and t+**o** is also free. Perform the flippingin fig. 2(a).
- **2** t is free and t+**o** is the first element of a block. Perform the flipping shown in fig. 2(b).
- **3** t is free but both t+1 and t-1 are the last elements of blocks. Perform the sequence of flippings shown in fig. 2(c).

- **4** t is in a block, t+**o** is free. Perform the flipping shown in fig. 2(d).
- **5** t is in a block, t+**o** is the first element of a block.Perform the flipping in fig. 2(e).
- **6** t is in a block with last element t+k***o**(k>0), t-**o** is the last element of another block and t+(k+1)***o** is free. Perform the sequence of flippings shown in fig 2(f) or 2(g) (depending on the relative position of the two blocks and t+(k+1)***o**.)

- **7** t is in a block with last element t+k***o**(k>0), t-**o** is the last element of another block and t+(k+1)***o** is ina block. Perform the sequence of flippings shown in fig 2(h) or 2(k) (depending on whether t+(k+1)***o** is at the beginning or at the end of its block.)
- **8** none of the above, σ has n-1 adjacencies; halt.

Proof of Correctness

Theorem: Algorithm (A) creates a permutation with n-1 adjacencies by at most (5n-7)/3 moves. **Proof:**

- If we have a permutation σ with less than n-1 adjacencies, on e of the firs seven cases is applicable. Thus, the algorithm does not stop unless n-1 adjacencies have been created.
- The algorithm will terminate, because at each iteration of the main loop, at least one new adjacency is created and none is destroyed. It remains to prove that the algorithm takes at most (5n-7)/3 moves.

Call:
action of case 1 -> action 1
action of case 2 -> action 2
action of case 3 & 6 -> action 3
action of case 4 -> action 4
action of case 5 -> action 5
action of case 7 -> action 7
Then, total number of moves is given by:

$$z = x_1 + x_2 + 4x_3 + x_4 + 2x_5 + x_7$$

The top of the stack before the flipping and the element next to t-**o** may be either:

1. be non-adjacent
2. form a new block
3. merge a block with a singleton
4. merge two blocks

We distinguosh among those cases by adding another index to x_3. So the total number of adjacencies in the conclusion of the algorithm is:

$$n - 1 = a + x_1 + x_2 + 2x_31 + 3x_32 + 3x_33 + 3x_34 + x_4 + x_5 + x_7$$

(1)

Also, if we define b as the number of blocks in π, because each move increases or decreases the number of blocks as in table 2, then:

$$b + x_1 - x_31 - x_33 - 2x_34 - x_5 - x_7 = 1$$

(2)
Because $b \leq a$, (1) becomes:

$$x_1 + x_2 + 2x_31 + 3x_32 + 3x_33 + 3x_34 + x_4 + x_5 + x_7 + b \leq n - 1$$

They maximize z subject to (2) and (3) and they show from duality by exhibiting a feasbile solution to the dual and primal problem, and get the bound (5n-7)/3.

Let $\tau = 17536428$, and let τ_k denote the sequence $1_k 7_k 5_k 3_k 6_k 4_k 2_k 8_k$, where $m_k = m + (k-1) * 8$
Consider the permutation $\chi = \tau_1 \tau_2^R \tau_3 \tau_4^R ... \tau_{m-1} \tau_m^R$, where m is an even integer, and n=|χ|=8*m.
Theorem: $19n/16 \leq f(x) \leq 17n/16$. **Scetch of Proof:** χ -> $\tau_2 \tau_1^R \tau_3$ ... -> $\tau_2^R \tau_1^R \tau_3$ ... -> $\tau_1 \tau_2 \tau_3$
Thus in 3n/16 moves, we get $1_k 2_k 3_k 4_k 5_k 6_k 7_k 8_k$.Also, each τ can be ordered in 8 steps, thus the upper bound established.

k-stable: a move is k-stable if it contains a substring of the form $1_k 7_k \sigma 2_k 8_k$ (or its reverse) **event:** χ_j is an event if χ_{j-1} is k-stable for some k, but χ_j, χ_{j+1},...,$\chi_{f(\chi)}$ are not.
Claim 1: There are exactly m events. **Claim 2:** $f(\chi) \geq n + w$.
Claim 3: $\forall j$, $1 \leq j \leq m - 1$, there exists a waste χ_l with $i_j \leq l \leq i_{j+1}$
Thus $f(\chi) \geq n + w \geq n + \frac{m}{2} = 17n/16$.

- They showed $3n/2-1 \leq f(n) \leq 2n+3$.

- Sorting pancakes *seems* to be $\mathcal{NP}$-Complete
- Sorting by *exchanging* is easy – Selection Sort
- Caprara: Sorting by reversals is $\mathcal{NP}$-Complete.
- Sorting n pancakes which can take only k different sizes is:
  - Easy for $k = 2$
  - For $k > 2$?
- Heydari and Sudborough: 3SAT $\leq_p$ PSP

PSP:

Instance	:	A permutation π of $1, 2, \ldots, n, n \geq 1$.
Question	:	Is it possible to transform π into a permutation π'*such* that $\pi'(1) = n$ by a sequence of prefix reversals each creating an adjacency without destroying any existing adjacencies? ($\pi'(1) = n$ means that the first element is n.)

M.H. Heydari, I.H. Sudborough, *Pancake Sorting is $\mathcal{NP}$-Complete*, working paper, 1995.

Decision Variables:

y_i^t	Depth of pancake i after flip t. $i = 1, \ldots, n; t = 1, \ldots, 2n$
x_t	The number of pancakes flipped from the top at flip t.
w_i^t	$\begin{cases} 1 & \text{if pancake } i \text{ is directly over pancake } i+1 \text{ after flip } t. \\ 0 & \text{Otherwise} \end{cases}$
z_t	$\begin{cases} 1 & \text{if pancakes are in order after flip } t. \\ 0 & \text{Otherwise} \end{cases}$
δ_i^t	$\begin{cases} 1 & \text{if pancake } i \text{ was flipped in flip } t. \\ 0 & \text{Otherwise} \end{cases}$
θ_i^t	Yet another binary variable to help model

Objective Function:

$$\min 2n - \sum_{i=0}^{2n} z^t + 1$$

$$\max \sum_{i=0}^{2n} z^t$$

Subject to:

$$\begin{aligned}
z_t &\leq w_i^t, \quad i = 1, \ldots n, t = 0, \ldots, 2n \\
z_t &\leq z_{t+1}, \quad t = 0, \ldots, 2n \\
x_t - y_i^t &\leq M\delta_i^t, \quad i = 0, \ldots, n-1, t = 1, \ldots, 2n \\
x_t - y_i^t &\geq -M(1-\delta_i^t), \quad i = 0, \ldots, n-1, t = 1, \ldots, 2n
\end{aligned}$$

$$y_i^t = \delta_i^t (x_t - y_i^{t-1} + 1) + (1 - \delta_i^t) y_i^{t-1}, \quad i = 0, \ldots, n-1, t = 1, \ldots, 2n$$

$$(\delta_i^t = 0 \Rightarrow y_i^t = y_i^{t-1})$$

$$\begin{aligned} y_i^t &\leq y_i^{t-1} + M\delta_i^t \\ y_i^t &\geq y_i^{t-1} - M\delta_i^t \end{aligned}$$

$$(\delta_i^t = 1 \Rightarrow y_i^t = x^t - y_i^{t-1} - 1)$$

$$\begin{aligned} y_i^t &\leq x^t - y_i^{t-1} - 1 + M\delta_i^t \\ y_i^t &\geq x^t - y_i^{t-1} - 1 - M\delta_i^t \end{aligned}$$

$$y_i^t = \delta_i^t(x_t - y_i^{t-1} + 1) + (1 - \delta_i^t)y_i^{t-1}, \quad i = 0, \ldots, n-1, t = 1, \ldots, 2n$$

$$(\delta_i^t = 0 \Rightarrow y_i^t = y_i^{t-1})$$

$$\begin{aligned} y_i^t &\leq y_i^{t-1} + M\delta_i^t \\ y_i^t &\geq y_i^{t-1} - M\delta_i^t \end{aligned}$$

$$(\delta_i^t = 1 \Rightarrow y_i^t = x^t - y_i^{t-1} - 1)$$

$$\begin{aligned} y_i^t &\leq x^t - y_i^{t-1} - 1 + M\delta_i^t \\ y_i^t &\geq x^t - y_i^{t-1} - 1 - M\delta_i^t \end{aligned}$$

$$(w_i^t = 1 \Rightarrow y_i^t = y_{i-1}^t + 1)$$

$$\begin{aligned} y_i^t &\leq y_{i-1}^t + 1 + M(1 - w_i^t) \\ y_i^t &\geq y_{i-1}^t + 1 - M(1 - w_i^t) \end{aligned}$$

$$(w_i^t = 0 \Rightarrow y_i^t \neq y_{i-1}^t + 1)$$

$$\begin{aligned} y_i^t - y_{i-1}^t &\leq M_1 w_i^t + M_2 \theta_i^t \\ y_i^t - y_{i-1}^t &\geq 2 - M_1 w_i^t - M_2(1 - \theta_i^t) \end{aligned}$$

- More than $8n^2$ constraints and $3n^2$ variables.
- Works O.K. with $n = 6, 7$
- Can solve specific instances of sizes $n = 8, 9, 10$
- Burns out for 11.

Decision Variables:

s_i^t : Size of pancake at depth i after flip t.

s_i^0 are given as input.$i = 1, \ldots, n; t = 0, \ldots, 2n$

x_t : The number of pancakes flipped from the top at flip t.

$$w_i^t : \begin{cases} 1 & \text{if size of pancake at depth } i \text{ is one less than the size of pancake at depth } i+1 \text{ after flip } t. \\ 0 & \text{Otherwise. } i = 1, \ldots, n-1; t = 0, \ldots, 2n \end{cases}$$

$$z^t : \begin{cases} 1 & \text{if pancakes are in order after flip } t. \\ 0 & \text{Otherwise. } t = 0, \ldots, 2n \end{cases}$$

$$\delta_i^t : \begin{cases} 1 & \text{if depth of flip} \geq i \text{ in flip } t. \\ 0 & \text{Otherwise. } i = 1, \ldots, n; t = 1, \ldots, 2n \end{cases}$$

$$e_{i,j}^t : \begin{cases} 1 & \text{if pancakes at depths } i,j \text{ were exchanged in flip } t. \\ 0 & \text{Otherwise. } i,j = 1, \ldots, n; t = 1, \ldots, 2n \end{cases}$$

$\theta_{i,j}^t, \beta_i^t$: More binary variables to help formulation

Objective Function:

$$\min 2n - \sum_{i=0}^{2n} z^t + 1$$

$$\max \sum_{i=0}^{2n} z^t$$

Subject to:
(stack is sorted at time t if each $w_i^t = 1 \quad \forall i$.)

$$\begin{aligned}
z^t &\leq w_i^t, \quad t = 0, \ldots, 2n; i = 1, \ldots, n-1 \\
z^t &\leq z^{t+1}, \quad t = 0, \ldots, 2n-1 \\
& (\delta_i^t = 1 \iff x^t \geq i) \\
x^t + M(1 - \delta_i^t) &\geq i, \quad i = 1, \ldots, n; t = 1, \ldots, 2n \\
x^t - M\delta_i^t &\leq i - 1, \quad i = 1, \ldots, n; t = 1, \ldots, 2n
\end{aligned}$$

$$(w_j^t = 1 \iff \text{size}(j) = \text{size}(j+1) + 1)$$

$$\begin{aligned}
s_j^t + 1 &\leq s_{j+1}^t + M(1 - w_j^t) \;\; j = 1, \ldots, n-1; t = 0, \ldots, 2n \\
s_j^t + 1 &\geq s_{j+1}^t - M(1 - w_j^t) \;\; j = 1, \ldots, n-1; t = 0, \ldots, 2n \\
s_j^t + 1 &\leq s_{j+1}^t - 1 + M(\beta_j^t + w_j^t) \;\; j = 1, \ldots, n-1; t = 0, \ldots, 2n \\
s_j^t &\geq s_{j+1}^t - M(1 - \beta_j^t + w_j^t) \;\; j = 1, \ldots, n-1; t = 0, \ldots, 2n
\end{aligned}$$

$$\begin{aligned}
(e_{i,j}^t &= 1 \iff i + j - 1 = x^t) \\
M_1\theta_{i,j}^t + M_2 e_{i,j}^t + i + j - 1 &\geq x^t + 1 \;\; i,j = 1, \ldots, n; t = 0, \ldots, 2n-1 \\
i + j - 1 &\leq x^t - 1 + M_1(1 - \theta_{i,j}^t) + M_2 e_{i,j}^t
\end{aligned}$$

$$(\delta_i^t = 0 \Rightarrow s_i^t = s_i^{t-1})$$

$$\begin{aligned} s_i^t &\leq s_i^{t-1} + M\delta_i^t, \quad i = 1, \dots, n; t = 1, \dots, 2n \\ s_i^t &\geq s_i^{t-1} - M\delta_i^t, \quad i = 1, \dots, n; t = 1, \dots, 2n \end{aligned}$$

$$(e_{i,j}^t = 1 \Rightarrow \text{positions } i,j \text{ are exchanged})$$

$$\begin{aligned} s_i^t &\leq s_j^{t-1} + M(1 - e_{i,j}^t), \quad i,j = 1, \dots, n; t = 1, \dots, 2n \\ s_i^t &\geq s_j^{t-1} - M(1 - e_{i,j}^t), \quad i,j = 1, \dots, n; t = 1, \dots, 2n - 1 \end{aligned}$$

- Exchange-based formulation.
- $\mathcal{O}(2n^3)$ variables and constraints.
- Preliminary model takes more time than model #1.
- Scope for a lot of improvement.