# Small Chvátal Rank

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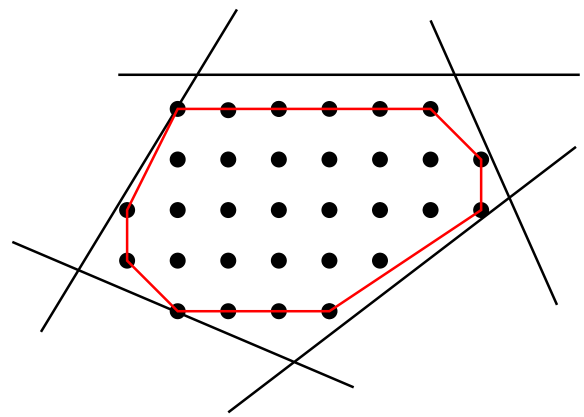
# The Integer Hull

Fix $A \in \mathbb{Z}^{m\times n}$, rank($A$) = $n$. For $\mathbf{b} \in \mathbb{Z}^m$, let

$Q_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$

$Q_{\mathbf{b}}^I :=$ conv$(Q_{\mathbf{b}} \cap \mathbb{Z}^n)$ **integer hull** of $Q_{\mathbf{b}}$



(1) $Q_{\mathbf{b}}^I$ is again a polyhedron.

(2) There is no function $f(m,n)$ that bounds #vertices($Q_{\mathbf{b}}^I$).

(Rubin 1970): $Q(k) := \{(x,y) \in \mathbb{R}^2_{\geq 0} : F_{2k}x + F_{2k+1}y \leq F_{2k+1}^2 - 1\}$

$F_k$ = $k$th Fibonacci number, $Q(k)^I$ has $k+3$ vertices (and edges)

(3) (Cook, Hartmann, Kannan, McDiarmid 1992):

size$(\mathbf{a}_i\mathbf{x} \leq b_i) \leq \phi \Rightarrow$ #vertices$(Q_{\mathbf{b}}^I) \leq 2m^n(6n^2\phi)^{n-1}$

(Barany, Howe, Lovász 1992): matching lower bound

**MAIN GOAL**: Given $A$, find $M \in \mathbb{Z}^{*\times n}$ such that for *each* $\mathbf{b} \in \mathbb{Z}^m$ there exists a $\mathbf{b}' \in \mathbb{Z}^*$ such that $Q^I_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : M\mathbf{x} \leq \mathbf{b}'\}$.

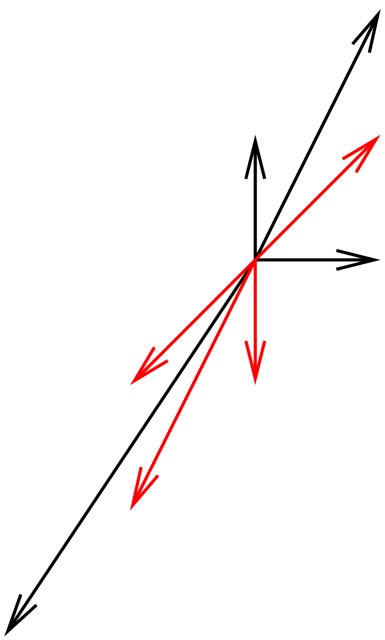
**Theorem 17.4** (Schrijver): Given $A$, such an $M$ exists.

**Proof idea**: $\Delta := \max |\text{subdet}(A)|$

Can cut out $Q^I_{\mathbf{b}}$ by $\mathbf{ax} \leq \beta$ where $||\mathbf{a}||_\infty \leq n^{2n}\Delta^n$

Set $\{\text{rows of } M\} = \{\mathbf{m} \in \mathbb{Z}^n : ||\mathbf{m}||_\infty \leq n^{2n}\Delta^n,\ \mathbf{m} \in \text{cone}(\text{rows of } A)\}$.

**Ex**: $A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Delta = 3, \quad n^{2n}\Delta^n = 144$



In fact, it is necessary and sufficient! to augment $A$ by $(1,1), (0,-1), (-1,-2), (-1,-1)$ to get $M$.

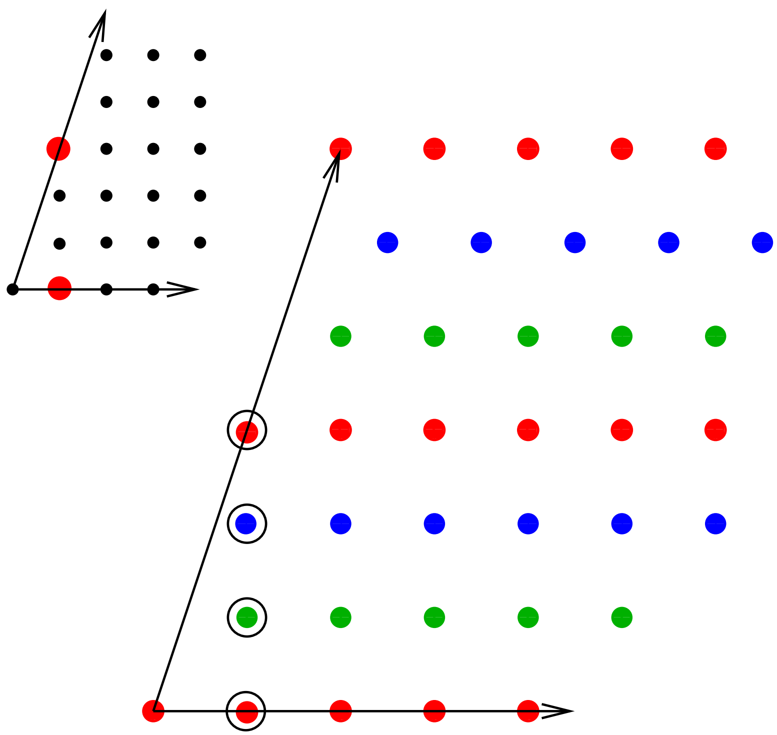
# Hilbert Bases

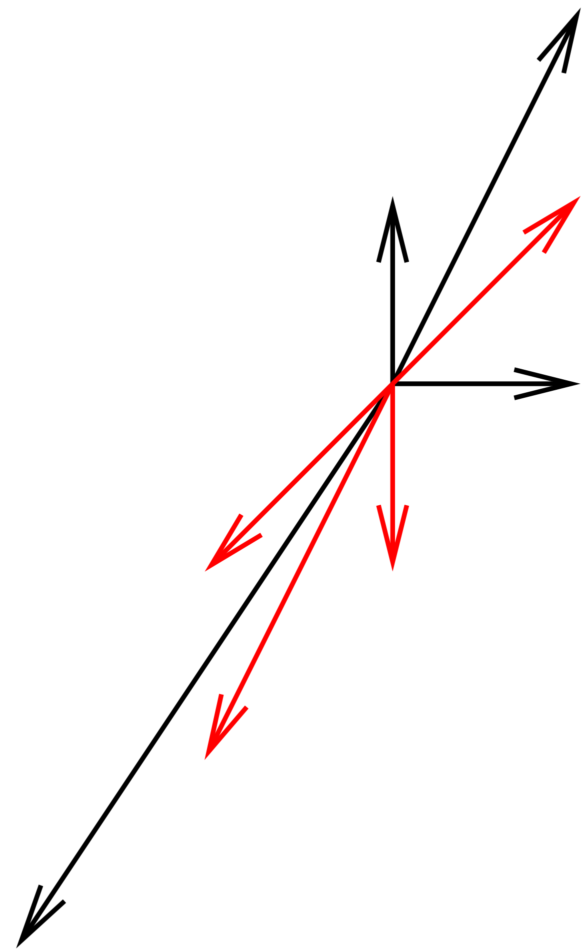
$K$ rational polyhedral cone

$\{\mathbf{h}_1, \ldots, \mathbf{h}_t\} \subset K$ is a **Hilbert basis**

of $K$ if $\forall\, \mathbf{u} \in K \cap \mathbb{Z}^n$, there exists

$\lambda_i \in \mathbb{N}$ such that $\mathbf{u} = \sum_{i=1}^{t} \lambda_i \mathbf{h}_i$.

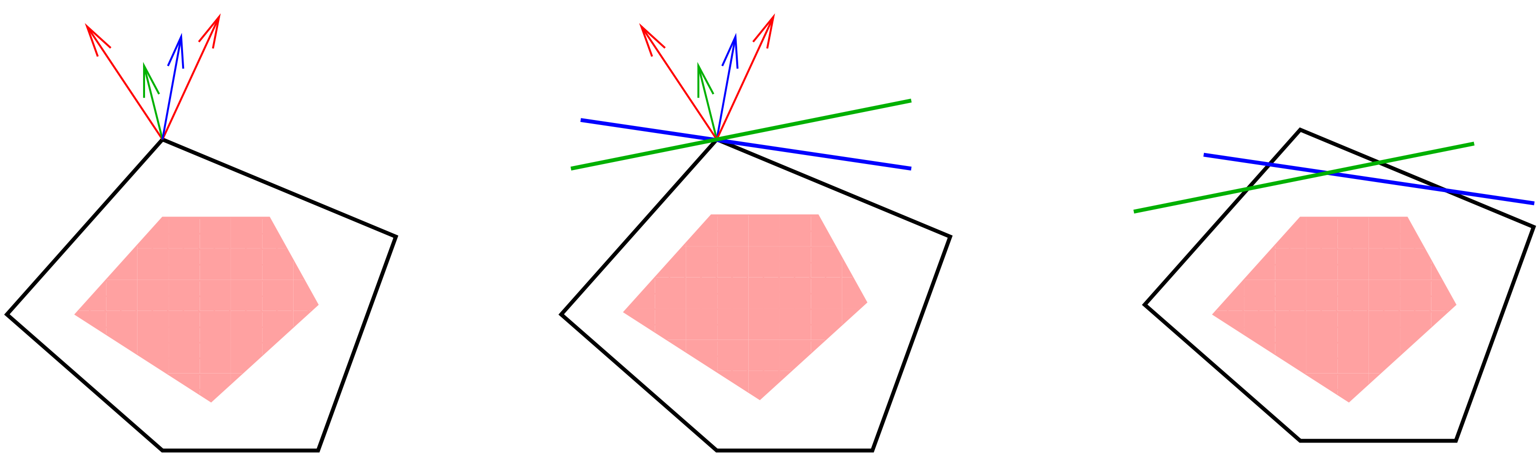




**Previous Example**

## Chvátal procedure

- $\mathcal{A} := \{\text{rows of } A\} \subset \mathbb{Z}^n$
- $\forall$ vertex $\mathbf{v}$ of $Q_{\mathbf{b}}$, let $\mathcal{A}_{\mathbf{v}} := \{\mathbf{a}_i \in \mathcal{A} : \mathbf{a}_i\mathbf{v} = b_i\}$
- $\mathbf{h} \in \text{HB}(\text{cone}(\mathcal{A}_{\mathbf{v}})) \Rightarrow \mathbf{hx} \leq \lfloor h\mathbf{v} \rfloor$ is valid for $Q^I_{\mathbf{b}}$
- $Q^{(1)}_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{R}^n : h\mathbf{x} \leq \lfloor h\mathbf{v} \rfloor \; \forall \, \mathbf{v}$ vertex of $Q_{\mathbf{b}}, \, \mathbf{h} \in \text{HB}(\text{cone}(\mathcal{A}_{\mathbf{v}}))\}$
- $Q^{(i+1)}_{\mathbf{b}} := (Q^{(i)}_{\mathbf{b}})^{(1)}$



# Chvátal Ranks

(Chvátal 1973, Schrijver 1980): There exists $t$ such that $Q^I_{\mathbf{b}} = Q^{(t)}_{\mathbf{b}}$.

**Definition**:

- Chvátal rank of $A\mathbf{x} \leq \mathbf{b}$ = min $\{t : Q^I_{\mathbf{b}} = Q^{(t)}_{\mathbf{b}}\}$
- Chvátal rank of $A$ = max $\{$ Chvátal rank $A\mathbf{x} \leq \mathbf{b} \; : \; \mathbf{b} \in \mathbb{Z}^m \}$

**Theorem 23.4** (Schrijver): Chvátal rank of $A$ is finite.

## Iterated Basis Normalization (IBN)

(1) Set $\mathcal{A}^{(0)} := \mathcal{A}$

(2) for $k \geq 1$, let $\mathcal{A}^{(k)} := \bigcup\{\text{HB}(\text{cone}(\mathcal{A}_\sigma^{(k-1)})) : \mathcal{A}_\sigma^{(k-1)}$ basis$\}$

(3) Stop if $\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)}$

**Lemma**:

- When $n = 2$, $\mathcal{A}^{(1)} = \mathcal{A}^{(2)}$ and IBN stops in at most two iterations
- IBN may not terminate when $n \geq 3$

**Key Fact**: Vectors generated by IBN contain normals of all inequalities created in the Chvátal procedure.

Let $A^{(k)}$ be a matrix with rows the vectors in $\mathcal{A}^{(k)}$.

**MAIN DEFINITIONS**:

(1) Small Chvátal rank (SCR) of ($A\mathbf{x} \leq \mathbf{b}$) := min $k$ such that $Q^I_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : A^{(k)}\mathbf{x} \leq \mathbf{b}'\}$ for some integral $\mathbf{b}'$.

(2) SCR$(A)$ $:=$ max$\{$SCR$(A\mathbf{x} \leq \mathbf{b})\ :\ \mathbf{b} \in \mathbb{Z}^m\}$.

**Proposition**:

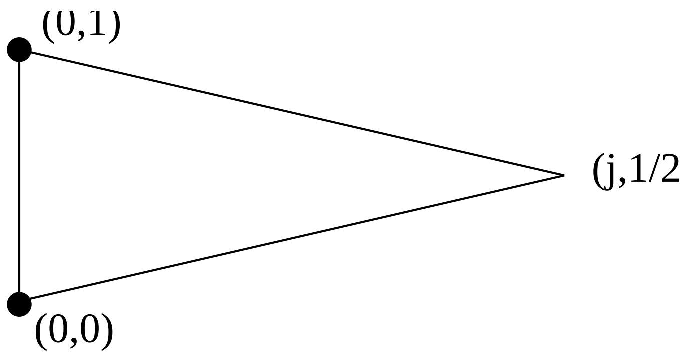
- SCR$(Ax \leq \mathbf{b}) \leq$ Chvátal rank ($A\mathbf{x} \leq \mathbf{b}$)
- SCR$(A) \leq$ Chvátal rank ($A$)

**Corollary**: SCR is finite.

## Example I: $n = 2$

$$n = 2 \Rightarrow \mathcal{A}^{(1)} = \mathcal{A}^{(2)} \Rightarrow \text{SCR}(A) \le 1.$$

When $n = 2$, Chvátal rank can be arbitrarily high!




Chvátal rank $\ge j$

**Theorem** (BT) For any $n \ge 2$ and $m \ge n+1$, there are systems $A\mathbf{x} \le \mathbf{b}$ with $A \in \mathbb{Z}^{m\times n}$ whose SCRs are one but Chvátal ranks are arbitrarily high.

## Example II: Stable set polytope of $K_n$

$\text{STAB}(K_n) = \text{conv}(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ is the integer hull of

$$Q(K_n) := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{ll} x_v \geq 0 & v \in V(K_n) \\ x_v + x_w \leq 1 & vw \in E(K_n) \end{array} \right\}$$

Only missing normal is $\mathbf{e} := (1, 1, \ldots, 1) \in \mathbb{R}^n$

(Chvátal 1973): Chvátal rank $(Q(K_n)) = O(\log n)$.

**Theorem** (BT): SCR $(Q(K_n)) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

**Example III: More stable set polytopes** (Annie Raymond 2007)

**Defn**: depth($\alpha \mathbf{x} \leq \beta$) = min $k$ s.t. $\alpha \in \mathcal{A}^{(k)}$.

**Theorem**: For any graph $G$, and $Q(G)$ as before,

(i) depth(clique inequality) $\leq 2$,

(ii) depth(odd-cycle inequality) $\leq 1$,

(iii) depth(odd-antihole inequality) $\leq 2$,

(iv) depth(odd-wheel inequality) $\leq 2$,

**Corollary**:

(1) SCR $(Q(G)) \leq 2$ if $G$ is a perfect graph. (i)

(2) SCR $(Q(G)) \leq 1$ if $G$ is a $t$-perfect graph. (ii)

(3) SCR $(Q(G)) \leq 2$ if $G$ is a $h$-perfect graph. (i),(ii)

# Chvátal rank = $0$

**Defn**: $\mathcal{A}$ is unimodular if $\forall\ \mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{A}'$ is a Hilbert basis for cone$(\mathcal{A}')$.

**Ex**: $U$ = vertex-edge incidence matrix of a bipartite graph

$U$ totally unimodular matrix & $\mathcal{A} = \{$rows of $U^t\}$ unimodular

**Theorem** (well known): The following are equivalent:

(1) $\mathcal{A}$ unimodular

(2) Every basis in $\mathcal{A}$ is a basis for $\mathbb{Z}^n$

(3) Every triangulation of $\mathcal{A}$ is unimodular

(4) $\forall \mathbf{b} \in \mathbb{Z}^m$, $Q_{\mathbf{b}}$ is integral

(5) Chvátal rank $(A) = 0$

# Supernormal Vector Configurations

**Defn** (Hoşten-Maclagan-Sturmfels 2004): $\mathcal{A}$ is supernormal if $\forall\ \mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{A} \cap \text{cone}(\mathcal{A}')$ is a Hilbert basis for $\text{cone}(\mathcal{A}')$.

**Ex**: $A = \left(\begin{array}{cccccc} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ \hline 1 & 1 & 1 & \cdots & 1 & 1 \end{array}\right)$ $\mathcal{A}$ supernormal, not unimodular

top part = edge-vertex incidence matrix of an odd cycle

# SCR = $0$

**Defn**: $A\mathbf{x} \leq \mathbf{b}$ is tight if for each $i = 1, \ldots, m$, $\mathbf{a}_i\mathbf{x} = b_i$ contains an integer point in $Q_{\mathbf{b}}$.

**Theorem** (BT): Let $\mathcal{A}$ consist of primitive vectors. Then the following are equivalent:

(1) $\mathcal{A}$ supernormal

(2) Every basis $\mathcal{A}' \subseteq \mathcal{A}$ has the property that $\mathcal{A} \cap \text{cone}(\mathcal{A}')$ is a Hilbert basis of $\text{cone}(\mathcal{A}')$

(3) Every triangulation of $\mathcal{A}$ that uses all the vectors is unimodular

(4) $\forall \mathbf{b} \in \mathbb{Z}^m$, $Q_{\mathbf{b}}$ is integral whenever tight

(5) SCR $(A) = 0$

# Lower bounds on SCR

**Theorem**(BT): For $m = n = 3$ (extends to $m \geq n \geq 3$), SCR($A\mathbf{x} \leq \mathbf{b}$) can be arbitrarily large and can grow exponentially in the size of the input.

**Ex**: $A_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & j & 2j-1 \end{pmatrix}$ $\quad j \geq 2$, integer $\Rightarrow$ SCR$(A_j) = j - 1$

**Proof**:

(1) $\mathcal{A}_j^{(j-1)} = \mathcal{A}_j^{(j)} \Rightarrow$ SCR$(A_j) \leq j - 1$

(2) $(1, j, j)$ is a facet normal for $\mathbf{b} = (0, 0, j-1)^t$

$(1, j, j) \in \mathcal{A}_j^{(j-1)} \backslash \mathcal{A}_j^{(j-2)}$

# Polytopes in the unit cube: Chvátal rank

$P \subseteq [0,1]^n$ polytope in the unit cube

Every $0,1$-polytope in $[0,1]^n$ has a linear relaxation in $[0,1]^n$

(Eisenbrand-Schulz 2003):

(1) The Chvátal rank of $P \subseteq [0,1]^n$ is $O(n^2 \log n)$.

(2) There are $P \subseteq [0,1]^n$ with Chvátal rank at least $(1+\varepsilon)n$.

Compare with convexification procedures in the $0,1$-case by

Adams-Sherali, Lovász-Schrijver, Lasserre that takes $n$ steps

**Conjecture**(Pokutta-Schulz-T.): SCR($P \subseteq [0,1]^n$) $\leq n$.

## Polytopes in the unit cube: SCR

**Theorem**(BT): For each $n$, there are systems $A\mathbf{x} \leq \mathbf{b}$ with $Q_{\mathbf{b}} \subset [0,1]^n$ with SCR at least $\frac{n}{2} - o(n)$.

**Proof idea**:

(1) (Alon-Vũ 1997, Ziegler 2000): $\exists\ 0,1$-polytope in $\mathbb{R}^n$ with facet normal $\mathbf{v}$ with $||\mathbf{v}||_\infty \geq \frac{(n-1)^{\frac{n-1}{2}}}{2^{2n+o(n)}}$

(2) every $0,1$-polytope has a linear relaxation in $[0,1]^n$ with facet normals in $\{-1,0,1\}^n$

(3) $\mathbf{v} \in \text{HB}(\text{cone}(\mathbf{v}_1,\ldots,\mathbf{v}_n)) \ \Rightarrow\ ||\mathbf{v}||_\infty \leq n \cdot \max||\mathbf{v}_i||_\infty$

(4) It takes $\geq \frac{n}{2} - o(n)$ iterations of IBN to generate the $\mathbf{v}$ in (1)

## Open Questions

(1) How does computing integer hulls with IBN compare in practice to existing methods?

(2) What is the complexity of checking supernormality?

(3) Characterize $A$ with SCR$(A) = k$, $k > 0$.

(4) Is there a better definition of SCR / different algorithm from IBN that generates facet normals of integer hulls? How to compute SCR($A$) in general?