On Mixed-Integer Quadratic Programming with Box Constraints

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[1] some joint work with Sam Burer, University of Iowa

Outline

Introduction

A *Mixed-Integer Quadratic Program with Box Constraints* (MIQPB) is a problem of the form:

$$\min \left\{ c^T x + x^T Q x : \; l \leq x \leq u, \; x_i \in \mathbb{R} \; (i \in C), \; x_i \in \mathbb{Z} \; (i \in I) \right\},$$

where $c \in \mathbb{Z}^n$, $Q \in \mathbb{Z}^{n \times n}$, $l \in \mathbb{Z}^n$ and $u \in \mathbb{Z}^n$.

We consider the (very difficult) case in which the objective is permitted to be non-convex.

Introduction (cont.)

MIQPB has two well-known (and $\mathcal{NP}$-hard) special cases:

- When all variables are constrained to be binary, we have *Unconstrained Boolean Quadratic Programming* (UBQP).
- When all variables are continuous, we have *Non-Convex Quadratic Programming with Box Constraints* (QPB).

UBQP is a classical problem in *combinatorial optimization*, but QPB is a classical problem in *global optimization*.

Introduction (cont.)

Why look at (non-convex) MIQPB?

- Most papers on MINLP focus on the convex case.
- Existing software for non-convex MINLP (e.g., BARON) can cope only with tiny instances.
- To tackle non-convex MINLP properly, we will need to combine MIP techniques with global optimization techniques.
- Non-convex MIQPB is a good place to start.

Introduction (cont.)

What am I actually doing?

- I started by taking known polyhedral results for UBQP and adapting them to QPB (joint work with Sam Burer).
- The convex sets associated with QPB turned out to be *much more complicated* than the polytopes associated with UBQP.
- Now I'm looking at the general mixed-integer case, and things are even more complicated!

The all-binary case: UBQP

There is a huge literature on UBQP. Some selected facts:

- Equivalent to *max-cut problem* (folklore).
- Thus, strongly $\mathcal{NP}$-hard (Garey *et al.*, 1976).
- A few polynomial cases known.
- People have looked at LP, CQP, SOCP and SDP relaxations.
- SDP approach is current winner (Rendl *et al.*, 2007).

The all-binary case (cont.)

The associated family of polytopes was introduced by Padberg:

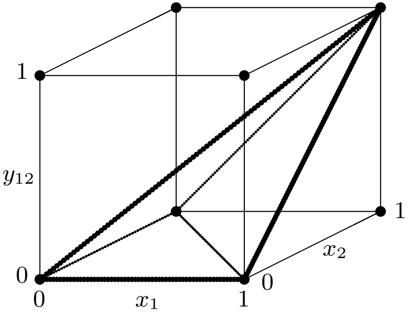
Definition (Padberg, 1989)

The *boolean quadric polytope* BQP_n is:

$$\text{conv}\left\{(x, y) \in \{0,1\}^{n+\binom{n}{2}} : \; y_{ij} = x_i x_j \; (1 \leq i < j \leq n)\right\}.$$

Here, y_{ij} is a new binary variable representing the product $x_i x_j$.
(No need to define y_{ii}, since $x_i^2 = 0$ when x_i binary.)

The all-binary case (cont.)



The all-binary case (cont.)

- Padberg (1989) introduced facet-inducing inequalities, called triangle, clique and cut inequalities.
- Other inequalities were found by Sherali *et al.* (1995), Boros & Hammer (1991,1993)...
- Even more can be derived from known results on the cut polytope (Deza & Laurent, 1997).
- But a complete description is known only for $n \leq 7$.

The all-continuous case: QPB

There is also a huge literature on QPB. Some facts:

- UBQP can be reduced to concave QPB (folklore).
- So QPB (continuous) is 'harder' than UBQP (discrete)!
- People have looked at LP and SDP relaxations.
- Traditional method is 'branch-and-reduce' (Tawarmalani & Sahinidis).
- But there are SDP approaches (Burer & Vandenbussche, 2007).

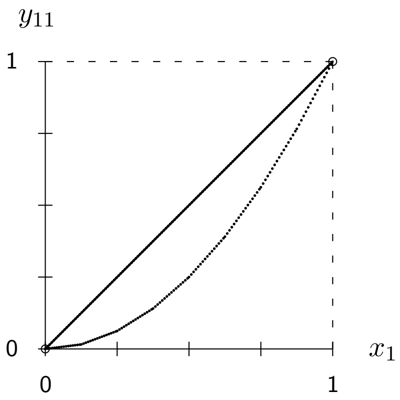
The all-continuous case (cont.)

We can assume $l_i = 0$ and $u_i = 1$ for all i. So the associated convex set is:

$$QPB_n = \text{conv}\left\{(x, y) \in [0, 1]^{n+\binom{n+1}{2}} : \ y_{ij} = x_i x_j \ (1 \leq i \leq j \leq n)\right\}.$$

As before, y_{ij} represents $x_i x_j$. (We now need to define y_{ii} as well.)

The all-continuous case (cont.)



The all-continuous case (cont.)

- Some simple inequalities can be derived from the *Reformulation-Linearization Technique* of Sherali & Adams.
- More inequalities can be derived from fact that $\binom{1}{x}\binom{1}{x}^T$ is psd (Shor).
- Yajima & Fujie (1998) showed that Padberg's clique and cut inequalities are valid for QPB_n.
- Anstreicher & Burer (2007) showed that the RLT and psd inequalities give a complete description for $n = 2$ (not trivial!).

The all-continuous case (cont.)

Burer & L. (2008) give several new results:

- RLT, clique and cut inequalities induce facets.
- Psd inequalities induce maximal faces.
- *All* valid inequalities for BQP_n are valid for QPB_n.
- But not every BQP facet yields a QPB facet.

Yet we still couldn't get a complete description for $n = 3$!

The all-integer case: IQPB

Now let's move on to the all-integer case ($C = \emptyset$).

- There is no literature.
- Strongly $\mathcal{NP}$-hard even in convex, unconstrained case. (Easy reduction from UBQP or CVP)
- Complexity status unknown even when $n = 2$. (But trivial to solve in pseudo-polynomial time.)
- Can assume $l_i = 0$ for all i.

The all-integer case (cont.)

Proposition

If

$$\sum_{i=1}^{n} \alpha_i x_i + \sum_{1 \le i \le j \le n} \beta_{ij} y_{ij} \le \gamma$$

is valid for QPB_n, then the 'stretched' inequality

$$\sum_{i=1}^{n} \frac{\alpha_i}{u_i} x_i + \sum_{1 \le i \le j \le n} \frac{\beta_{ij}}{u_i u_j} y_{ij} \le \gamma$$

is valid for $IQPB(n, u)$.

The all-integer case (cont.)

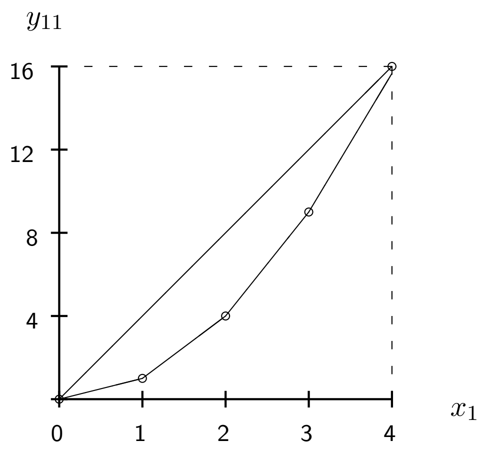
Conjecture

If an inequality induces a facet of QPB_n, then the stretched inequality induces a facet of $IQPB(n, u)$.

(Easy to prove if the inequality induces a facet of BQP_n as well.)

In any case, stretched inequalities are not enough even when $n = 1$...

The all-integer case (cont.)



The all-integer case (cont.)

To make progress, we use *split disjunctions* of the form:

$$(v^T x \leq s) \vee (v^T x \geq s + 1)$$

where $v \in \mathbb{Z}^n$ and $s \in \mathbb{Z}$. These imply:

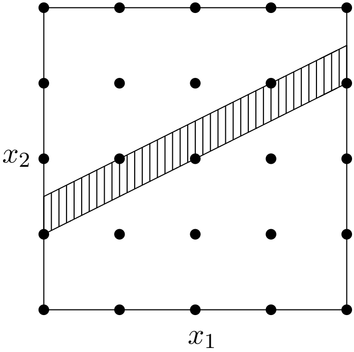
$$(v^T x - s)(v^T x - s - 1) \geq 0.$$

From this we obtain 'split' inequalities of the form:

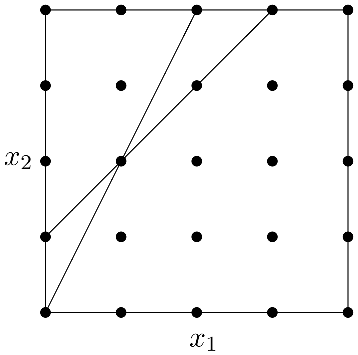
$$\sum_{i=1}^{n} v_i^2 y_{ii} + \sum_{1 \leq i < j \leq n} v_i v_j y_{ij} - (2s) v^T x + s(s+1) \geq 0.$$

Gives complete description for $n = 1$. But not for $n = 2$!

The all-integer case: standard 'split'



The all-integer case: non-standard 'split'



The all-integer case (cont.)

These non-standard splits yield expressions of the form:

$$(a^T x - b)(c^T x - d) \geq 0.$$

Linearising, we obtain new facets of $IQPB(n, u)$.

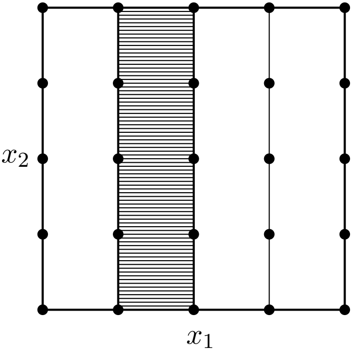
According to PORTA, there are even more facets when $n = 2$!

The general case: MIQPB

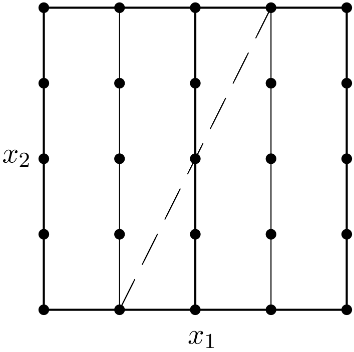
Finally, we have the MIQPB itself.

- We get all of the ‘stretched’ inequalities.
- The ‘split’ inequalities are still valid provided $v_i = 0$ for all $i \in C$.
- The ‘non-standard split’ inequalities are still valid provided $a_i = 0$ for all $i \in C$.

The general case: standard 'split'



The general case: non-standard 'split'



Summary

- We understand BQP_n quite well, and QPB_n reasonably well.
- But $IQPB(n, u)$ and $MIQPB(n, u)$ are extremely complex, even for $n = 2$.
- An important open question: can IQPB or MIQPB be solved in polynomial time when $n = 2$?
- If so, can we get a complete description for $n = 2$?

One Last Remark

Results on MIQPB can be applied to general MIQPs! Here's how:

- Suppose our constraints are $Ax \leq b$, $l \leq x \leq u$.
- Add slack variables to yield $Ax + Is = b$.
- Compute upper bounds $s \leq u'$ (e.g., by solving LPs or IPs).
- Decide whether slacks are continuous or integer.
- Derive valid inequalities for $l \leq x \leq u$, $0 \leq s \leq u'$.
- Project back to original space.

Does this give a new (stronger) version of the Sherali-Adams and Lovász-Schrijver operators?