Mingling: Mixed-Integer Rounding with Bounds

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Joint work with Alper Atamtürk @ U. C. Berkeley

$$S \;=\; \left\{(x,s) \in \mathbb{Z}^n \times \mathbb{R} \,:\, \sum_{i\in N} a_i x_i + s \geq b,\; x \geq 0\;, s \geq 0\right\}$$

MIR inequaity for S

$$\sum_{i\in N}\left(\hat{b}\lfloor a_i\rfloor + \min\{\hat{b},\hat{a}_i\}\right)x_i + s \geq \hat{b}\lceil b\rceil \qquad \text{where} \qquad \hat{c} = c - \lfloor c\rfloor$$

α-MIR ($\alpha > 0$)

$$\sum_{i\in N}\mu_{\alpha,b}(a_i)x_i + s \geq \mu_{\alpha,b}(b)$$

where

$$\mu_{\alpha,b}(c) = r_b\lfloor c/\alpha\rfloor + \min\{r_b, r_c\},\; \text{and}\; r_c = c - \alpha\lfloor c/\alpha\rfloor$$

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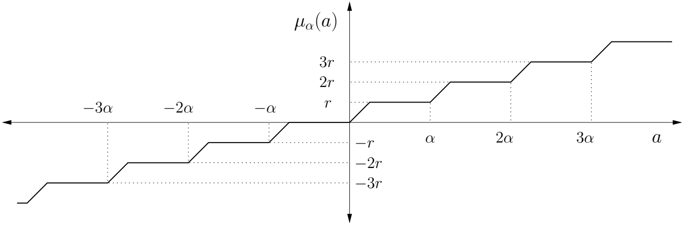
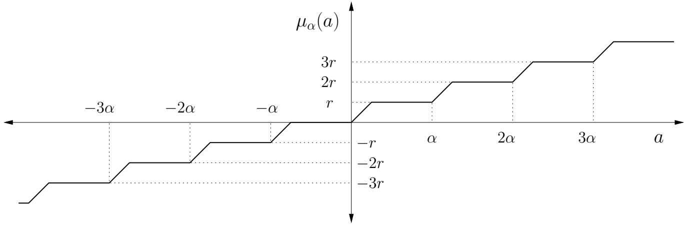
For $x, s \geq 0$:
$$\sum_{i \in N} a_i x_i + s \geq b \quad \Rightarrow \quad \sum_{i \in N} \mu_{\alpha,b}(a_i) x_i + s \geq \mu_{\alpha,b}(b)$$



Property: MIR function $\mu_{\alpha,b}$ is nondecreasing and subadditive.

MIR cuts: Nemhauser & Wolsey, 88, 90
Split cuts: Cook, Kannan, Schrijver, 90
Gomory mixed integer cuts: Gomory, 60
Aggregate, complement & α-MIR: Marchand & Wolsey, 01

For $x, s \geq 0$: $$\sum_{i \in N} a_i x_i + s \geq b \quad \Rightarrow \quad \sum_{i \in N} \mu_{\alpha,b}(a_i) x_i + s \geq \mu_{\alpha,b}(b)$$



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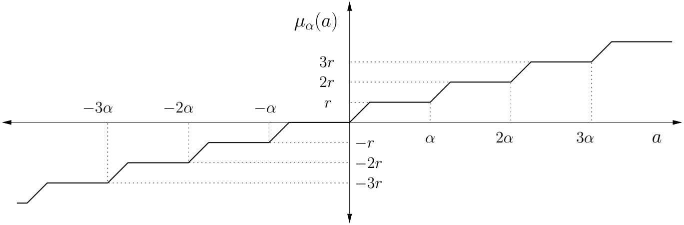

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Consider a simple example

$$S = \left\{ x \in \mathbb{Z}^2, s \in \mathbb{R} \; : \; -5x_1 + x_2 + s \geq 0.5,\; x_1 \geq 0,\; 2 \geq x_2 \geq 0,\; s \geq 0 \right\}$$

The MIR inequality for S

$$-2.5x_1 + 0.5x_2 + s \geq 0.5$$

is actually valid for $S^R = \{x \in \mathbb{Z}^2, s \in \mathbb{R} \; : \; -5x_1 + x_2 + s \geq 0.5,\; s \geq 0\}$

It is dominated by

$$-4.0x_1 + 0.5x_2 + s \geq 0.5$$

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Let

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where $a_1 < 0 < b < 1 \leq u_2$.

The following inequality is valid for S:

$$(a_1 + (1-b)u_2)x_1 + bx_2 + s \geq b.$$

Proof: Rewrite base inequality

$$(a_1 + u_2)x_1 + (x_2 - u_2 x_1) + s \geq b.$$

Disjunction: $x_1 = 0$ and $x_1 \geq 1$

$$(a_1 + u_2)x_1 + b(x_2 - u_2 x_1) + s \geq b$$

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Disjunction: $x_1 = 0$

$$+ b(x_2 \qquad\qquad) + s \geq b$$

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In addition, MIR inequality for S is

$$\min\{a_1 - \lfloor a_1 \rfloor, b\}x_1 + b(x_2 + \lfloor a_1 \rfloor x_1) + s \geq b$$

Proposition

$$min\{a_1 + k, b\}x_1 + b(x_2 - kx_1) + s \geq b$$

where $k := min\{u_2, -\lfloor a_1 \rfloor\}$ *is valid and facet defining for* $\mathrm{conv}(S)$

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Complement x_2?

$$a_1 x_1 - (u_2 - x_2) + s \geq b - u_2$$

MIR inequality is

$$(\min\{a_1 - \lfloor a_1 \rfloor, b\} + \lfloor a_1 \rfloor) x_1 - b(u_2 - x_2) + s \geq b(1 - u_2)$$

or,

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The new inequality is really new.

Consider the set

$$K_{\geq} := \left\{ x \in \mathbb{Z}^N, s \in \mathbb{R} : \sum_{i \in I} a_i x_i + \sum_{j \in J} a_j x_j + s \geq b,\ u \geq x \geq 0,\ s \geq 0 \right\}$$

where $b \geq 0$ and $a_i > 0$ for $i \in I$ and $a_j < 0$ for $j \in J$ $\ (I \cup J = N)$

Let the mingling set $I^+ = \{1, \ldots, n\} \subseteq \{i \in I : a_i > b\}$ be such that $a_1 \geq a_2 \geq \ldots \geq a_n$

For $j \in J$, let the mingling set of j be $I_j = \{1, \ldots, t\}$ *"minimally cover"* a_j:

$$\sum_{i=1}^{t-1} a_i u_i + a_t \bar{u}_{tj} \geq |a_j| \quad \text{where } u_j \geq u_{tj} \geq 1$$

(if $\sum_{i=1}^{n} a_i u_i < |a_j|$ then let $I_j = I^+$)

For $i \in I$, define $J_i := \{j \in J : i \in I_j\}$.

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Mingling

Base inequality

$$\sum_{i \in I^+} a_i x_i + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{j \in J} a_j x_j + s \geq b$$

Base inequality mingled

$$\sum_{i \in I^+} a_i (x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{j \in J} (a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}) x_j + s \geq b$$

Mingling inequality

$$\sum_{i \in I^+} b (x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{j \in J} \min\{b, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}\} x_j + s \geq b$$

Proposition. Mingling inequality is valid for $K_{\geq}$. It is facet-defining if

$$b - \min\{a_j + \kappa : j \in \bar{J}\} \geq \min\{a_i : a_i > b, \ i \in I \setminus I^+\}$$

$\kappa := \sum_{i \in I^+} a_i u_i$ and $\bar{J} := \{j \in J : a_j + \kappa < 0\}$.

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Symmetric mingling inequalities

Consider sets

$$\mathcal{K}_{\geq} = \left\{ x \in \mathbb{Z}^N, s \in \mathbb{R} : \ ax + s \geq b, \ u \geq x \geq 0, \ s \geq 0 \right\}$$

$$\mathcal{K}_{\leq} = \left\{ x \in \mathbb{Z}^N, s' \in \mathbb{R} : \ ax \leq b + s', \ u \geq x \geq 0, \ s' \geq 0 \right\}$$

Lemma. Inequality $\pi x + s \geq \pi_o$ is valid for $\mathcal{K}_{\geq}$ if and only if

$$(a - \pi)x \leq b - \pi_o + s' \qquad (**)$$

is valid for $\mathcal{K}_{\leq}$. Moreover, it defines a facet of $\mathsf{conv}(\mathcal{K}_{\geq})$ if and only if inequality (**) defines a facet of $\mathsf{conv}(\mathcal{K}_{\leq})$.

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Lemma. Inequality $\pi x + s \geq \pi_o$ is valid for $\mathcal{K}_{\geq}$ if and only if

$$(a - \pi)x \leq b - \pi_o + s' \qquad (**)$$

is valid for $\mathcal{K}_{\leq}$. Moreover, it defines a facet of $\mathsf{conv}(\mathcal{K}_{\geq})$ if and only if inequality (**) defines a facet of $\mathsf{conv}(\mathcal{K}_{\leq})$.

Mingling inequalities are defined for the set

$$\mathcal{K}_{\geq} = \left\{ x \in \mathbb{Z}^N, s \in \mathbb{R} : \; ax + s \geq b, \; u \geq x \geq 0, \; s \geq 0 \right\}$$

when $b \geq 0$.

If $b \leq 0$, consider

$$\mathcal{K}'_{\geq} = \left\{ x \in \mathbb{Z}^N, s \in \mathbb{R} : \; -ax \leq -b + s, \; u \geq x \geq 0, \; s \geq 0 \right\}$$

for which valid inequalities can be obtained from

$$\mathcal{K}'_{\leq} = \left\{ x \in \mathbb{Z}^N, s \in \mathbb{R} : \; -ax + s' \geq -b, \; u \geq x \geq 0, \; s \geq 0 \right\}$$

Assumption: $b \leq 0$. Let $J^- \subseteq \{j \in J : a_j < b\}$.

Base

$$\sum_{i \in I} a_i x_i + \sum_{j \in J^-} a_j x_j + \sum_{j \in J \setminus J^-} a_j x_j + s \geq b$$

Base mingled

$$\sum_{i \in I} (a_i + \sum_{j \in J_i} a_j \bar{u}_{ji}) x_i + \sum_{j \in J^-} a_j (x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i) + \sum_{j \in J \setminus J^-} a_j x_j + s \geq b$$

Symmetric mingling inequality

$$\sum_{i \in I} \min\{a_i + \sum_{j \in J_i} a_j \bar{u}_{ji} - b, 0\} x_i + \sum_{j \in J^-} (a_j - b)(x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i) + s \geq 0$$

Assumption: $b \leq 0$. Let $J^- \subseteq \{j \in J : a_j < b\}$.

Base

$$\sum_{i\in I} a_i x_i + \sum_{j\in J^-} a_j x_j + \sum_{j\in J\setminus J^-} a_j x_j + s \geq b$$

Base mingled

$$\sum_{i\in I}(a_i + \sum_{j\in J_i} a_j \bar{u}_{ji})x_i + \sum_{j\in J^-} a_j(x_j - \sum_{i\in I_j} \bar{u}_{ji} x_i) + \sum_{j\in J\setminus J^-} a_j x_j + s \geq b$$

Symmetric mingling inequality

$$\sum_{i\in I} \min\{a_i + \sum_{j\in J_i} a_j \bar{u}_{ji} - b, 0\} x_i + \sum_{j\in J^-} (a_j - b)(x_j - \sum_{i\in I_j} \bar{u}_{ji} x_i) + s \geq 0$$

Base inequality mingled

$$\sum_{i\in I^+} a_i(x_i - \sum_{j\in J_i} \bar{u}_{ij}x_j) + \sum_{i\in I\setminus I^+} a_ix_i + \sum_{j\in J}(a_j + \sum_{i\in I_j} a_i\bar{u}_{ij})x_j + s \geq b$$

Mingling inequality

$$\sum_{i\in I^+} b(x_i - \sum_{j\in J_i} \bar{u}_{ij}x_j) + \sum_{i\in I\setminus I^+} a_ix_i + \sum_{j\in J}\min\{b, a_j + \sum_{i\in I_j} a_i\bar{u}_{ij}\}x_j + s \geq b$$

Two-step mingling inequality

$$\sum_{i\in I^+} \mu_{\alpha,b}(b)(x_i - \sum_{j\in J_i} \bar{u}_{ij}x_j) + \sum_{i\in I\setminus I^+} \mu_{\alpha,b}(a_i)x_i + \sum_{j\in J} \mu_{\alpha,b}(\min\{b, a_j + \sum_{i\in I_j} a_i\bar{u}_{ij}\})x_j + s \geq \mu_{\alpha,b}(b)$$

where $\alpha > 0$ st $\alpha\lceil b/\alpha\rceil \leq \min\{a_i : i \in I^+\}$

Proposition. Two-step mingling inequality is valid for $\mathcal{K}_{\geq}$. It defines a facet of $\mathsf{conv}(K_{\geq})$ if $\bar{J} = \emptyset$, $I^+ = \{i \in I : a_i \geq \alpha\lceil b/\alpha\rceil\}$, $\alpha = a_i$ for some $i \in I$.

Base inequality mingled

$$\sum_{i\in I^+} a_i(x_i - \sum_{j\in J_i} \bar{u}_{ij}x_j) + \sum_{i\in I\setminus I^+} a_i x_i + \sum_{j\in J}(a_j + \sum_{i\in I_j} a_i\bar{u}_{ij})x_j + s \geq b$$

Mingling inequality

$$\sum_{i\in I^+} b(x_i - \sum_{j\in J_i} \bar{u}_{ij}x_j) + \sum_{i\in I\setminus I^+} a_i x_i + \sum_{j\in J}\min\{b, a_j + \sum_{i\in I_j} a_i\bar{u}_{ij}\}x_j + s \geq b$$

Two-step mingling inequality

$$\sum_{i\in I^+} \mu_{\alpha,b}(b)(x_i - \sum_{j\in J_i} \bar{u}_{ij}x_j) + \sum_{i\in I\setminus I^+} \mu_{\alpha,b}(a_i)x_i + \sum_{j\in J}\mu_{\alpha,b}(\min\{b, a_j + \sum_{i\in I_j} a_i\bar{u}_{ij}\})x_j + s \geq \mu_{\alpha,b}(b)$$

where $\alpha > 0$ st $\alpha\lceil b/\alpha\rceil \leq \min\{a_i : i \in I^+\}$

Proposition. Two-step mingling inequality is valid for $\mathcal{K}_\geq$. It defines a facet of $\mathsf{conv}(K_\geq)$ if $\bar{J} = \emptyset$, $I^+ = \{i \in I : a_i \geq \alpha\lceil b/\alpha\rceil\}$, $\alpha = a_i$ for some $i \in I$.

Symmetric two-step mingling inequality

$$\sum_{j\in J^-}(a_j+\mu_{\alpha,b}(-b))(x_j-\sum_{i\in I_j}\bar{u}_{ji}x_i)+\sum_{j\in J\setminus J^-}(a_j+\mu_{\alpha,b}(-a_j))x_j$$
$$+\sum_{i\in I}(a_i+\sum_{j\in J_i}\bar{u}_{ji}+\mu_{\alpha,b}(\min\{-b,-a_i-\sum_{j\in J_i}a_j\bar{u}_{ji}\}))x_i+s\geq b+\mu_{\alpha,b}(-b)$$

where $\alpha > 0$ such that $\min\{a_j : j \in J^-\} \leq \alpha\lfloor b/\alpha\rfloor$

Proposition. Symmetric two-step mingling inequality is valid for $\mathcal{K}_\geq$. It defines a facet of $\mathsf{conv}(K_\geq)$ if $\bar{I}=\emptyset$, $J^-=\{j\in J : a_j\leq\alpha\lfloor b/\alpha\rfloor\}$, $\alpha=a_j$ for some $j\in J$.

Application: 0-1 variables

$$K^1_{\leq} := \left\{(x,s) \in \{0,1\}^N \times \mathbb{R} \,:\, \sum_{i \in N} a_i x_i \leq b + s,\ s \geq 0\right\} \quad (\text{wlog } a > 0)$$

$C \subseteq N$ is a *cover* if $\lambda := \sum_{i \in C} a_i - b > 0$

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{i \in N \setminus C} -a_i x_i + s \geq \lambda$$

For $I^+ \subseteq \{i \in C : a_i > \lambda\}$ the mingling inequality is

$$\sum_{i \in I^+} \lambda(\bar{x}_i - \sum_{j \in J_i} x_j) + \sum_{i \in C \setminus I^+} a_i \bar{x}_i + \sum_{j \in N \setminus C} \min\{\lambda, -a_j + \sum_{i \in J_i} a_i\} x_j + s \geq \lambda$$

If $I^+ = \{i \in C : a_i > \lambda\}$

$$\sum_{i \in C} \min\{\lambda, a_i\} \bar{x}_i + \sum_{j \in N \setminus C} (-\lambda |I_j| + \min\{\lambda, -a_j + \sum_{i \in J_i} a_i\}) x_j + s \geq \lambda$$

Continuous cover inequality (Marchand & Wolsey, '99).

Application: 0-1 variables

$$K_{\leq}^{1} := \left\{(x,s) \in \{0,1\}^N \times \mathbb{R} \,:\, \sum_{i \in N} a_i x_i \leq b + s,\ s \geq 0\right\} \quad (\text{wlog } a > 0)$$

$C \subseteq N$ is a *cover* if $\lambda := \sum_{i \in C} a_i - b > 0$

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{i \in N \setminus C} -a_i x_i + s \geq \lambda$$

For $I^+ \subseteq \{i \in C : a_i > \lambda\}$ the mingling inequality is

$$\sum_{i \in I^+} \lambda(\bar{x}_i - \sum_{j \in J_i} x_j) + \sum_{i \in C \setminus I^+} a_i \bar{x}_i + \sum_{j \in N \setminus C} \min\{\lambda, -a_j + \sum_{i \in J_i} a_i\} x_j + s \geq \lambda$$

If $I^+ = \{i \in C : a_i > \lambda\}$

$$\sum_{i \in C} \min\{\lambda, a_i\} \bar{x}_i + \sum_{j \in N \setminus C} (-\lambda |I_j| + \min\{\lambda, -a_j + \sum_{i \in J_i} a_i\}) x_j + s \geq \lambda$$

Continuous cover inequality (Marchand & Wolsey, '99).

$$K^1_{\leq} := \left\{(x,s) \in \{0,1\}^N \times \mathbb{R} \;:\; \sum_{i \in N} a_i x_i \leq b + s,\; s \geq 0\right\} \quad \text{(wlog } a > 0\text{)}$$

$C \subseteq N$ is a *cover* if $\lambda := \sum_{i \in C} a_i - b > 0$

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{i \in N \setminus C} -a_i x_i + s \geq \lambda$$

For $I^+ \subseteq \{i \in C : a_i > \lambda\}$ the mingling inequality is

$$\sum_{i \in I^+} \lambda(\bar{x}_i - \sum_{j \in J_i} x_j) + \sum_{i \in C \setminus I^+} a_i \bar{x}_i + \sum_{j \in N \setminus C} \min\{\lambda, -a_j + \sum_{i \in J_i} a_i\} x_j + s \geq \lambda$$

If $I^+ = \{i \in C : a_i > \lambda\}$

$$\sum_{i \in C} \min\{\lambda, a_i\} \bar{x}_i + \sum_{j \in N \setminus C} (-\lambda |I_j| + \min\{\lambda, -a_j + \sum_{i \in J_i} a_i\}) x_j + s \geq \lambda$$

Continuous cover inequality (Marchand & Wolsey, '99).

Assume $\sum_{i \in C \setminus k} a_i < b$ for some $k \in C$. Alternatively,

$$\sum_{i \in C \setminus k} a_i \bar{x}_i + \sum_{i \in N \setminus (C \setminus k)} -a_i x_i + s \geq \lambda - a_k =: -\theta < 0$$

For $J^- \subseteq \{j \in N \setminus (C \setminus k) : a_j > \theta\}$ and $I = C \setminus k$, symmetric mingling inequality

$$\sum_{i \in C \setminus k} \min\{0, a_i - \sum_{j \in J_i} a_j + \theta\} \bar{x}_i + \sum_{j \in J^-} (\theta - a_j)(x_j - \sum_{i \in I_j} \bar{x}_i) + s \geq 0$$

If $J^- = \{j \in N \setminus (C \setminus k) : a_j > \theta\}$

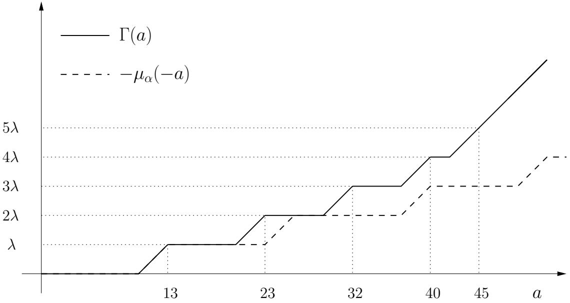
$$\sum_{i \in C \setminus k} (\min\{0, a_i - \sum_{j \in J_i} a_j + \theta\} + \sum_{j \in J_i} (a_j - \theta)) \bar{x}_i - \sum_{j \in N \setminus (C \setminus k)} (a_j - \theta)^+ x_j + s \geq 0$$

Reverse continuous cover inequality (Marchand & Wolsey, 99).

Example

$$13x_1 + 10x_2 + 9x_3 + 8x_4 + 5x_5 + ax_6 \le 42 + s,\ x \in \{0,1\}^6,\ s \ge 0.$$

Cover $C = \{1,2,3,4,5\}$, $\lambda = 13 + 10 + 9 + 8 + 5 - 42 = 3$ and $\bar{a} = 13$.



$$\sum_{i=1}^{5} \lambda x_i + \underbrace{\left(4\lambda - \min\{\lambda, -a + \sum_{i=1}^{4} a_i\}\right)}_{\Gamma(a)} x_6 \le 4\lambda + s$$

(Gu, Nemhauser & Savelsbergh, '99)
(Marchand & Wolsey, '99)

$$K_{\leq}^{u} := \left\{(x,s) \in \mathbb{Z}^N \times \mathbb{R} : \sum_{i \in N} a_i x_i \leq b + s,\ u \geq x \geq 0,\ s \geq 0\right\}, (\text{wlog } a > 0)$$

$C \subseteq N$ is a *cover* if $\lambda := \sum_{i \in C} a_i u_i - b > 0$

Base inequality

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{i \in N \setminus C} -a_i x_i + s \geq \lambda$$

For $I^+ \subseteq \{i \in C : a_i > \lambda\}$ and $J = N \setminus C$, mingling inequality

$$\sum_{i \in I^+} \lambda(\bar{x}_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{i \in C \setminus I^+} a_i \bar{x}_i + \sum_{j \in J} \min\{\lambda, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}\} x_j + s \geq \lambda$$

For $I^+ \subseteq \{i \in C : a_i \geq a_k \lceil \lambda / a_k \rceil\}$, two-step inequality with $\alpha = a_k$

$$\sum_{i \in I^+} (u_k - \eta + 1)(a_k - r)(\bar{x}_i - \sum_{j \in J_i} x_j) + \sum_{i \in C \setminus I^+} \mu_{a_k}(a_i) \bar{x}_i$$
$$+ \sum_{j \in J} \mu_{a_k}(\min\{\lambda, -a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}\}) x_j + s \geq (u_k - \eta + 1)(a_k - r)$$

Application: Bounded integers

Assume $\sum_{i \in C \setminus k} u_i a_i < b$ for some $k \in C$. Alternatively

$$\sum_{i \in C \setminus k} a_i \bar{x}_i + \sum_{i \in N \setminus (C \setminus k)} -a_i x_i + s \geq \lambda - a_k u_k =: -\theta < 0$$

For $J^- \subseteq \{j \in N \setminus (C \setminus k) : a_j > \theta\}$ and $I = C \setminus k$, symmetric mingling inequality

$$\sum_{i \in C \setminus k} \min\{0, a_i - \sum_{j \in J_i} a_j \bar{u}_{ji} + \theta\} \bar{x}_i + \sum_{j \in J^-} (\mu - a_j)(x_j - \sum_{i \in I_j} \bar{u}_{ji} \bar{x}_i) + s \geq 0$$

For $J^- \subseteq \{j \in N \setminus (C \setminus k) : a_j \geq a_k \lceil \theta / a_k \rceil\}$, symmetric two-step inequality with $\alpha = a_k$

$$\sum_{j \in J^-} (-a_j + \eta r)(x_j - \sum_{i \in I_j} \bar{u}_{ji} \bar{x}_i) + \sum_{j \in J \setminus J^-} (-a_j + \mu_{a_k}(a_j)) x_j$$
$$+ \sum_{i \in C \setminus k} (a_i + \sum_{j \in J_i} \bar{u}_{ji} + \mu_{a_k}(\min\{\theta, -a_i + \sum_{j \in J_i} a_j \bar{u}_{ji}\})) \bar{x}_i + s \geq -\theta + \eta r$$

Application: Bounded integers

Atamtürk's continuous integer knapsack cover inequalities

$$\sum_{i \in C} -\Phi_k(-a_i)\bar{x}_i + \sum_{j \in N \setminus C} -\gamma_k(a_j)x_j + s \geq (u_k - \eta + 1)(a_k - r)$$

and continuous integer knapsack pack inequalities

$$\sum_{j \in N \setminus (C \setminus k)} -\Phi_k(a_j)x_j + \sum_{i \in C \setminus k} -\omega_k(-a_i)\bar{x}_i + s \geq -\theta + \eta r$$

where

$$\Phi_k(a) = \begin{cases} (\eta - u_k - 1)(a_k - r) & \text{if } a < -\lambda, \\ a - \mu_{a_k}(a) & \text{if } -\lambda \leq a \leq \theta, \\ a - \eta r & \text{if } a > \theta, \end{cases} \quad \text{for } k \in N$$

and γ_k and ω_k are (complicated) superadditive lifting functions are in fact two-step and symmetric two-step mingling inequalities.

$\min\{r, \epsilon\}$ gives a feasible solution with objective value $\min\{r, \epsilon\}$ larger than for $(z(a_1), y(a_1))$, contradicting its optimality.

(ii) $z_\ell(a_1) = 0$ and $z_\ell(a_2) < u_\ell - \eta$: Let $\kappa = u_\ell - \eta - z_\ell(a_2)$. Using $\varphi \leq r$, (29) gives

$$\begin{aligned} y(a_1) &> -\bar{a}_k - a_\ell(u_\ell - \eta - \kappa) - r \geq (a_\ell(\eta - u_\ell) - \bar{a}_k) + a_\ell \kappa - r \\ &\geq a_\ell \kappa - r \geq a_\ell - r. \end{aligned}$$

Thus again $y(a_1) = a_\ell - r + \epsilon$ with $\epsilon > 0$. Since $z_\ell(a_1) = 0$, in this case the solution obtained from $(z(a_1), y(a_1))$ by increasing $z_k(a_1)$ by one and increasing $z_\ell(a_1)$ by $u_\ell - \eta$ and $y(a_1)$ by $a_\ell - r + \min\{\epsilon, \delta_k + r\}$ is feasible and improves the objective value by $\min\{\epsilon, \delta_k + r\}$. If $\delta_k + r > 0$, this contradicts the optimality of $(z(a_1), y(a_1))$. If $\delta_k + r = 0$, we have an alternative solution in which $z_k(a_1)$ is one larger.

(iii) $z_\ell(a_1) = 0$ and $z_\ell(a_2) = u_\ell - \eta$: Using $\varphi \leq \delta_k + r$ from (29) we have

$$\begin{aligned} y(a_1) &> -\bar{a}_k - a_\ell u_\ell + a_\ell \eta - \delta_k - r \\ &= (-\bar{a}_k + (\eta - u_\ell - 1)a_\ell - \delta_k) + a_\ell - r = a_\ell - r. \end{aligned}$$

Since and $z_\ell(a_1) = 0$ and $y(a_1) = a_\ell - r + \epsilon$ with $\epsilon > 0$ the case reduces to case 2 above. □

Theorem 5. *If $\bar{a}_i \leq d - u_\ell a_\ell$ or $\bar{a}_i \geq r - a_\ell$ for all $i \in I^-$, then*

$$\Gamma(a) = \begin{cases} u_{ih}(u_\ell - \eta + 1)(a_\ell - r) & \text{if } m_{ih} \leq a \leq m_{ih} + \delta_i, \\ (u_{ih}(u_\ell - \eta + 1) + k)(a_\ell - r) & \text{if } m_{ih} + \delta_i + k a_\ell \leq a \leq m_{ih} + \delta_i + k a_\ell + r, \\ u_{ih}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{ih} - \delta_i - (k+1)r & \text{if } m_{ih} + \delta_i + k a_\ell + r \leq a \leq m_{ih} + \delta_i + (k+1)a_\ell, \\ (u_{su_s}(u_\ell - \eta + 1) + p)(a_\ell - r) & \text{if } m_{J^{--}} + p a_\ell \leq a \leq m_{J^{--}} + p a_\ell + r, \\ u_{su_s}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{J^{--}} - (p+1)r & \text{if } m_{J^{--}} + p a_\ell + r \leq a \leq m_{J^{--}} + (p+1)a_\ell, \\ (u_{iu_i}(u_\ell - \eta + 1) + \eta)(a_\ell - r) & \text{if } \bar{m}_{ih} \leq a \leq \bar{m}_{ih} + \delta_i + r \\ (u_{iu_i}(u_\ell - \eta + 1) + \eta)(a_\ell - r) + a - \bar{m}_{ih} - \delta_i - (k+1)r & \text{if } \bar{m}_{ih} + \delta_i + k a_\ell + r \leq a \leq \bar{m}_{ih} + \delta_i + (k+1)a_\ell \\ (u_{iu_i}(u_\ell - \eta + 1) + \eta + k)(a_\ell - r) & \text{if } \bar{m}_{ih} + \delta_i + k a_\ell \leq a \leq \bar{m}_{ih} + \delta_i + k a_\ell + r \\ u_{nu_n}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{I^{--}} - \eta r & \text{if } a \geq m_{I^{--}}, \end{cases}$$

where $\delta_i = (\eta - u_\ell - 1)a_\ell - \bar{a}_i$ for $i \in I^{--}$, $u_{ih} = \sum_{k=1}^{i-1} u_k + h$, $m_{ih} = m_{(i-1)u_{i-1}} - h\bar{a}_i$ for $h \in \{0, 1, \ldots, u_i\}$ and $i \in \{1, 2, \ldots, n\}$ with $m_{0u_0} = 0$, $\bar{m}_{ih} = m_{ih} + \eta a_\ell$ for $h \in \{0, 1, \ldots, u_i\}$ and $i \in \{s, s+1, \ldots, n\}$, $m_{J^{--}} = m_{su_s}$, $m_{I^{--}} = \bar{m}_{nu_n}$, $k \in \{0, 1, \ldots, u_\ell - \eta\}$, and $p \in \{0, 1, \ldots, \eta - 1\}$.

Proof. From Lemma 4 (ii), we may assume that $z_i = 0$ for all $i \in I^-$ such that $\bar{a}_i \geq r - a_\ell$. Hence the condition of Lemma 5 is satisfied. Observe that if $I^{--} = \emptyset$, inequality (26) equals (14). Consequently $\Gamma(a) = \Omega(a)$ for $a \in \mathbb{R}_+$ as $m_{J^{--}} = 0$ and $m_{I^{--}} = \eta a_\ell$. Otherwise, from Lemma 4 (i) and Lemma 5, as a increases, there exist

Concluding remarks

- Simple procedure; uses MIR functions only; easy to implement
- Combines MIR and superadditive lifting
- Produces strong inequalities even with bounds
- A new general way for deriving valid inequalities for MIPs
- Suggests new separation methods for high rank/lifted cuts