Computational testing of exact separation for mixed-integer knapsack problems

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(joint work with Maurizio Boccia* and Igor Vasiliev**)

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MIP 2008 - Columbia University

MIP solvers & cutting planes

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- Knapsack $\Rightarrow$ Lifted Cover Inequalities
- Mixed knapsack $\Rightarrow$ Mixed-Integer Rounding (MIR) inequalities
- Tableau rows $\Rightarrow$ Gomory Cuts

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- We can try to generate "cuts outside the template paradigm" (local cuts: Applegate, Bixby, Chvátal and Cook, 2000)
- Local cuts proved to be successful for the TSP
- Based on exact separation.

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- A separation algorithm is said exact if it either guarantees to provide a valid inequality for P cutting off $\bar{x}$ or concludes that $\bar{x} \in P$.

Exact separation of valid inequalities for the knaspack polytope

The knapsack set (Boyd, 1988)

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The exact separation LP *SEPLP*(X^K):

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\max \quad & \bar{\boldsymbol{y}}\boldsymbol{\pi} - \pi_0 & \\
& \boldsymbol{w\pi} \le \pi_0, \quad \boldsymbol{w} \in X^K & (1)\\
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Inequalities (1) ensure that the inequality is satisfied from every feasible solution in X^K.

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Let $\boldsymbol{\pi}^*, \pi_0^*$ be the optimal solution of $SEPLP(X^K)$.

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Extreme points of *SEPLP*(X^K) are in one-to-one correspondence with the facets of *conv*(X^K).

Recent results

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Generalized Assignment problem

- Medium-size Generalized Assignment instances *d*10200 and *d*20200 solved to optimality for the first time.
- Integrality gap reduced on many larger benchmark instances (up to 80*x*1600) (A., Boccia and Vasilyev, 2007).

Recent results (cont.)

Single Source Capacitated Facility Location Problems

- Reformulation based on dicut inequalities + exact separation (Boccia, 2007).
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Set Covering

- Exact separation for subsets of formulation constraints (A., Boccia and Vasyliev, 2007).
- *seymour* solved to optimality on a single workstation.

A step further: the mixed-integer knapsack set X^{MI}

We consider single-row mixed-integer knapsack relaxations of MIP problems:

$$X^{MI} = \{(\boldsymbol{y}, \boldsymbol{x}) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \ \ \boldsymbol{ay} + \boldsymbol{gx} \leq b, \ \boldsymbol{y} \leq \boldsymbol{u}, \boldsymbol{x} \leq \boldsymbol{v}\}$$

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- Atamturk (2002) studied the polyhedral structure of $conv(X^{MI})$.
- Fukasawa and Goycoolea (2007) proposed an exact separation routine for X^{MI}. The core of their separation procedure is a sophisticated Branch-and-Bound algorithm for the mixed-integer knapsack problem.

The knapsack set with a single continuous variable X^{MK}

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we remove bounds $\boldsymbol{v}$ and aggregate the continuous variables we get the "weaker" knapsack set with a single continuous variable X^{MK}:

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Why we focus on X^{MK}

The set X^{MK} is a better candidate for a "lightweight" exact separation routine.

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- They characterized several other classes of valid inequalities for $conv(X^{MK})$

Exact separation for $conv(X^{MK})$

Any valid inequality for $conv(X^{MK})$ has the form:

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with $\boldsymbol{\pi}$, σ and π_0 nonnegative.

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Extreme points of $SEPLP(X^{MK})$ are in one-to-one correspondence with the facets of $conv(X^{MK})$.

Solving $SEPLP(X^{MK})$ by row generation

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Step 2 Solve the *partial separation* problem $SEPLP(S)$:

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Let $(\boldsymbol{\pi}^*, \sigma^*, \bar{\pi}_0^*)$ be the optimal solution of $SEPLP(S)$.

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\begin{aligned}
\max \quad & \boldsymbol{\pi}^* \boldsymbol{w} - \bar{\sigma}^* t \\
& (\boldsymbol{w}, t) \in X^{MK}
\end{aligned}
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to check whether the "candidate inequality" $\boldsymbol{\pi}^* \boldsymbol{y} - \sigma^* \boldsymbol{s} \leq \pi_0^*$ is valid for $conv(X^{MK})$.

Solving $SEPLP(X^{MK})$ by row generation (cont.)

Step 4 Let $(\hat{\boldsymbol{w}}, \hat{t})$ be the optimal solution of *MKNAP*. If $\boldsymbol{\pi}^* \hat{\boldsymbol{w}} - \sigma^* \hat{t} > \pi_0^*$ then set $S = S \cup \{(\hat{\boldsymbol{w}}, \hat{t})\}$ and goto **Step 1**.

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Step 5 $(\boldsymbol{\pi}^*, \sigma^*, \pi_0^*)$ is the optimal solution of $SEPLP(X^{MK})$ and the inequality $\boldsymbol{\pi}^* \boldsymbol{y} - \sigma^* s \leq \pi_0^*$ is valid for $conv(X^{MK})$.

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It follows that:

$$(\hat{t} = 0) \vee (\hat{t} = \boldsymbol{a}\hat{\boldsymbol{w}} - b > 0)$$

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The optimal solution of MKNAP is the best between the optimal solutions of the two following knapsack problems:

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KNAP2 $(t = \mathbf{a}\mathbf{w} - b)$:

$$\begin{aligned} \min \quad & (\bar{\sigma}^* \mathbf{a} - \pi^*)\mathbf{w} \\ & \mathbf{a}\mathbf{w} \geq b + 1 \\ & \mathbf{w} \in \mathbb{Z}^n \end{aligned}$$

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Both the knapsack problems can be solved very fast by dynamic programming (Pisinger, 2004).

Implementation details

When embedded into a cutting plane algorithm, $SEPLP(X^{MK})$ is applied to each row defining a mixed-integer knapsack set:

$$X^{MI} = \{(\boldsymbol{y}, \boldsymbol{x}) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \ \ \boldsymbol{ay} + \boldsymbol{gx} \leq b, \ \boldsymbol{y} \leq \boldsymbol{u}, \boldsymbol{x} \leq \boldsymbol{v}\}.$$

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Preprocessing: transform the mixed integer set X^{MI} into the mixed-integer knapsack set X^{MK}.

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Preprocessing: transform the mixed integer set X^{MI} into the mixed-integer knapsack set X^{MK}.

Convert coefficients into integers (required to use dynamic programming)

Bound substitution

- Consider the mixed-integer set

$$X^{MI} = \{(\boldsymbol{y}, \boldsymbol{x}) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \ \ \boldsymbol{ay} + \boldsymbol{gx} \leq b, \ \boldsymbol{y} \leq \boldsymbol{u}, \boldsymbol{lx} \leq \boldsymbol{v}\}.$$

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- The MIP formulation can also include some additional variable bounds on the continuous variables.
- Bound substitution consists of replacing some continuous variables by their respective simple/variable bounds. It is done heuristically by performing one of the following substitutions:

$$x_j = l_j + x_j'; x_j = v_j - x_j'; x_j = \tilde{l}_j y_i + x_j'; w_j = \tilde{v}_j y_k - x_j'$$

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- Let $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{x}})$ be the current fractional solution. The bound with smallest slack is selected for substitution. That is, let

$$\mu = \min\{\bar{x}_j - l_j, \ v_j - \bar{x}_j, \ \bar{x}_j - \tilde{l}_j \bar{y}_i, \ \tilde{v}_j \bar{y}_k - \bar{x}_j\}.$$

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Preprocessing

Let

$$\sum_{i \in I} a'_i y_i + \sum_{j \in P} g'_j x'_j \leq b',$$

with $0 \leq y_i \leq u_i \; \forall j \in I$ and $x'_j \geq 0 \; \forall j \in P$, be the mixed-integer inequality after bound substitution.

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$$s = -\sum_{j \in P^-} g'_j x'_j,$$

where $P^- = \{j \in P : \; g'_j < 0\}$.

Preprocessing

Let

$$\sum_{i \in I} a_i' y_i + \sum_{j \in P} g_j' x_j' \leq b',$$

with $0 \leq y_i \leq u_i \ \forall j \in I$ and $x_j' \geq 0 \ \forall j \in P$, be the mixed-integer inequality after bound substitution.

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  $$s = -\sum_{j \in P^-} g_j' x_j',$$

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- All the integer variables with negative coefficients are complemented:

  $$y_j = \begin{cases} u_j - y_j' & \text{if } a_j' < 0 \\ y_j' & \text{otherwise} \end{cases}$$

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- If the procedure fails, we discard the inequality since too large coefficients may cause numerical problems.

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Computational results

- Computational experiments were carried out on a 64bit Pentium Quad-core 2.6 GHz processor with 4 Gb RAM. The LP solver was Xpress 2007B.

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- The test bed consists of all the MIPLIB 2003 mixed-integer instances and of the “Mittleman” instances *bc1, bienst1, bienst2, binkar10_1, dano3-4, dano3-5*. We set a limit of 300 CPU secs for the time spent in separation.

Computational results (cont.)

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- For simplicity of comparison, separation routines run on the original (i.e. not preprocessed) instances.

Computational results

Name	SCIP LB	SCIP %Gap	SCIP Time	*MK*-SEP LB	*MK*-SEP % Gap	*MK*-SEP Time
10teams	917.00	0.00	0.08	917	0.00	0.96
a1c1s1	997.53	0.00	0.14	997.53	0.00	2.20
aflow30a	983.16	0.00	0.00	1053.29	40.11	10.07
aflow40b	1005.50	0.00	0.03	1058.32	32.50	10.67
arki001	7579599.81	0.00	0.46	7579599.81	0.00	0.89
atlanta-ip	81.25	0.11	11.14	82.46	13.91	300.00
dano3mip	576.23	0.00	0.56	576.23	0.00	7.40
danoint	62.63	0.00	0.01	62.66	0.88	3.59
fiber	385094.10	91.66	0.27	390493.82	93.82	9.26
fixnet6	3192.04	71.57	0.09	3442.60	80.58	196.21
gesa2	25691081	71.28	0.44	25701859	74.86	4.29
gesa2-o	25476489	0.0	0.06	25588105	37.02	7.79
glass4	800002400	0.00	0.01	800002400	0.00	0.23
liu	385.00	4.92	0.64	385.00	4.92	8.76
markshare1	0.00	0.00	0.00	0.00	0.00	43.79
markshare2	0.00	0.00	0.00	0.00	0.00	26.82
mas74	10482.79	0.00	0.00	10482.79	0.00	0.25
mas76	38901.02	0.64	0.00	38901.02	0.64	0.16
misc07	1415.00	0.00	0.00	1415	0.00	0.54
mkc	-607.18	9.73	4.62	-605.83	12.54	56.40
modglob	20430947.60	0.00	0.02	20431515.90	0.18	9.20
msc98-ip	19538746.75	5.58	16.65	19559084.16	11.97	169.54
net12	31.55	7.27	7.97	32.08	7.54	106.53
nsrand-ipx	49851.43	41.87	4.92	49877.59	43.00	80.75
roll3000	12072.71	54.41	2.13	12073.49	54.46	23.06
swath	334.50	0.00	0.53	334.5	0.00	9.18
timtab1	195605.34	22.68	0.07	229628.78	27.30	3.23
timtab2	250004.21	16.43	0.16	270295.07	18.43	6.84
tr12-30	18124.17	3.36	0.01	84403.46	60.27	8.23
vpm2	10.40	13.21	0.02	11.21	33.94	1.59
binkar10_1	6701.56	61.42	1.33	6720.55	79.54	9.06
bienst1	11.72	0.00	0.01	14.01	6.54	2.15
bienst2	11.72	0.00	0.00	14.88	7.41	3.18
dano3-4	576.23	0.00	0.41	576.23	0.00	2.76
dano3-5	576.23	0.00	0.52	576.23	0.00	3.15
rgn	68.00	57.49	0.00	68.00	57.49	1.14

The overall effect

Some preliminary tests on non-trivial instances (Cplex 11.1)

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- **nsrand-ip** After 5000 B&B nodes: with the original formulation the gap is 1.5%. Using exact separation the relative gap is 0.37%.

Some considerations

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- Exact separation not applicable to large and dense rows.

Back to the template paradigm: Mixed Knapsack Inequalities

We focus on mixed knapsack inequalities (Marchand and Wolsey, 2002), which can described by the following procedure.
Given:

$$X^{BMK} = \{(\boldsymbol{y}, s) \in \mathbb{B}^n_+ \times \mathbb{R}_+ : \ \ \boldsymbol{a}\boldsymbol{y} - s \leq b, \ \boldsymbol{y} \leq \boldsymbol{u}\}$$

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ii) Find a valid inequality $\alpha\boldsymbol{y} \leq \beta$ for the resulting binary knapsack polytope;

$$X^{BMK}_{\bar{s}} = \{\boldsymbol{y} \in \mathbb{B}^n_+ : \ \ \boldsymbol{ay} \leq b + \bar{s}, \ \boldsymbol{y} \leq \boldsymbol{u}\}$$

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iii) lift the s to get a valid inequality for X_{BMK} of the form $\boldsymbol{\alpha y} - \gamma s \leq \beta$.

Mixed Knapsack Inequalities: lifting the s

Let

$$\eta(s) = \max \quad \boldsymbol{\alpha y} \tag{5}$$
$$\boldsymbol{ay} \leq b + s \tag{6}$$
$$\boldsymbol{y} \in \{0, 1\}^n \tag{7}$$

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Proposition

The inequality

$$\boldsymbol{\alpha y} \leq \beta + \gamma s$$

is valid for conv(X^{BMK}) *if* $\eta(s) \leq \beta + \gamma s$ *for each* $s \in \mathbb{R}_+$*.*

Mixed Knapsack Inequalities: lifting the s (cont.)

A geometrical interpretation

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Mixed Knapsack Inequalities: lifting the *s* (cont.)

A geometrical interpretation

- $\eta(s)$ is a step function
- the line $\beta + \gamma s$ is a "valid" rhs if it defines an upper bound on the $\eta(s)$, for each $s \in \mathbb{R}_+$.

Mixed Knapsack Inequalities: lifting the s (cont.)

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Step 1 Solve the problem:

$$
\begin{aligned}
\zeta = \quad \max \quad & \boldsymbol{\alpha y} - \gamma s \\
& \boldsymbol{\alpha y} \leq b + s \\
& \boldsymbol{y} \in \{0, 1\}^N \\
& s \geq 0
\end{aligned}
$$

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Step 2 If $\zeta \leq \beta$ then the inequality $\boldsymbol{\alpha y} \leq \beta + \gamma s$ is valid for $conv(X^{BMK})$. STOP.

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Step 3 Increase γ and Go to Step 1.

Mixed Knapsack Inequalities: lifting the s (cont.)

A numerical example

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- Consider the set $X^{BMK} = \{7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 - s \leq 6\}$.

Mixed Knapsack Inequalities: lifting the s (cont.)

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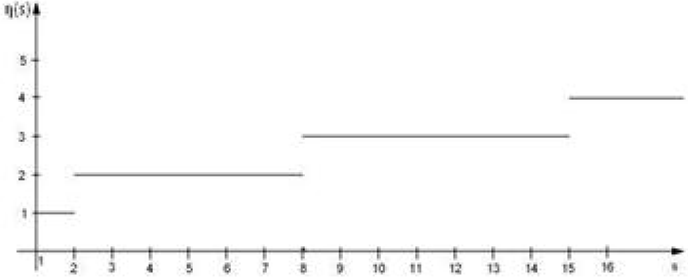
- $\eta(s)$ step function.



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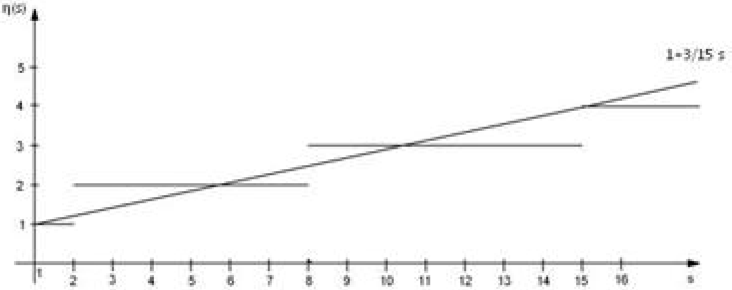
- Initialization: $\gamma = 3/15$.



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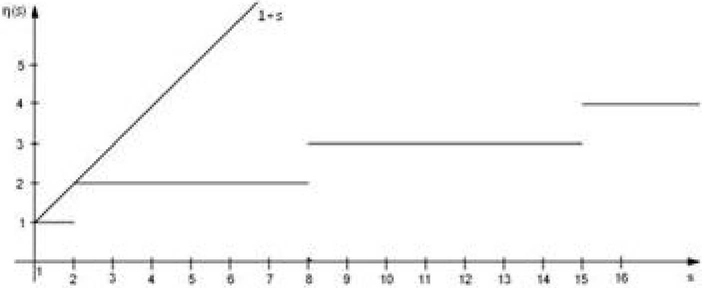
- Iteration 1: update $\gamma = 1$; $y_1 + y_2 + y_3 + y_4 - s \leq 1$ is valid.



Computational results for Mixed Knapsack Inequalities

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gesa2	25691081	71.28	0.44	25691081	71.28	0.11
gesa2-o	25476489	0.0	0.06	25476489	0.00	0.06
glass4	800002400	0.00	0.01	800002400	0.00	0.03
liu	385.00	4.92	0.64	385.00	4.92	0.64
markshare1	0.00	0.00	0.00	0.00	0.00	0.03
markshare2	0.00	0.00	0.00	0.00	0.00	0.01
mas74	10482.79	0.00	0.00	10482.79	0.00	0.02
mas76	38901.02	0.64	0.00	38901.02	0.64	0.08
misc07	1415.00	0.00	0.00	1415	0.00	0.01
mkc	-607.18	9.73	4.62	-611.48	0.77	1.61
modglob	20430947.60	0.00	0.02	20431458.13	0.16	0.05
msc98-ip	19538746.75	5.58	16.65	19557387.00	11.43	57.76
net12	31.55	7.27	7.97	31.91	7.45	37.92
nsrand-ipx	49851.43	41.87	4.92	49851.67	41,88	13.64
roll3000	12072.71	54.41	2.13	12072.71	54.41	0.17
swath	334.50	0.00	0.53	334.5	0.00	0.61
timtab1	195605.34	22.68	0.07	213136.28	25.06	0.03
timtab2	250004.21	16.43	0.16	250086.12	16.44	0.08
tr12-30	18124.17	3.36	0.01	84363.73	60.24	0.26
vpm2	10.40	13.21	0.02	11.31	36,79	0.08
binkar10_1	6701.56	61.42	1.33	6637.18	0.00	0.14
bienst1	11.72	0.00	0.01	14.03	6.59	0.22
bienst2	11.72	0.00	0.00	14.88	7.41	0.27
dano3-4	576.23	0.00	0.41	576.23	0.00	11.67
dano3-5	576.23	0.00	0.52	576.23	0.00	12.45
rgn	68.00	57.49	0.00	68.00	57.49	0.01

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- How to select MIP substructures to ensure that exact separation leads to violated cuts?