Two Submodular Optimization Problems On Risk Aversion

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Submodular utility function

Given a finite set N and $a, b \in \mathbb{R}^N$ consider set function $u : 2^N \to \mathbb{R}$ s.t.

$$u(S) := f(a(S)) + b(S), \quad S \subseteq N,$$

where f is strictly concave, increasing, differentiable function.

Notation: For $v \in \mathbb{R}^N$, $v(S) := \sum_{i \in S} v_i$ for $S \subseteq N$.

Definition

A set function $h : 2^N \to \mathbb{R}$ is *submodular* if for $S \subseteq N$ and $i \in N \setminus S$

$$\rho_i(S) := h(S \cup i) - h(S) \text{ is nonincreasing in } S.$$

Optimization problems

$$\max_{S \subseteq N} u(S)$$

$$\min_{S \subseteq N} u(S)$$

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VaR

Uncertain loss (negative return) ℓ_i ; $\quad x \in X \subseteq \{0,1\}^n \times [0,1]^m$
For $\ell_i \sim N(\mu_i, \sigma_i^2)$ and $\epsilon > 0$

$$\zeta(\epsilon) := \min \left\{ z : \mathbf{Prob} \left(\sum_i \ell_i x_i > z \right) \leq \epsilon, \ x \in X \right\}$$

$$\min \left\{ f(x) = \sum_i \mu_i x_i + \Omega(\epsilon) \sqrt{\sum_i \sigma_i^2 x_i^2} : x \in X \right\} \qquad (CIP)$$

$\ell_i \sim N(\mu_i, \sigma_i^2)$: $\Omega(\epsilon) = -\Phi^{-1}(\epsilon)$

Only first two moments μ_i, σ_i^2 known: $\Omega(\epsilon) = \sqrt{(1-\epsilon)/\epsilon}$
(Bertsimas and Popescu 05; El Ghaoui et al. 03)

Symmetric with support $[\mu_i - \sigma_i, \mu_i + \sigma_i]$: $\Omega(\epsilon) = \sqrt{-2 \ln \epsilon}$
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r_i: uncertain return with mean μ_i and variance σ_i^2

$$\zeta = \max \left\{ \mu x - \Omega(\epsilon) \sqrt{\sum_i \sigma_i^2 x_i^2} : \sum_i a_i x_i \leq b, x \in \{0,1\}^n \times [0,1]^m \right\}$$

where $\Omega(\epsilon) = \sqrt{(1-\epsilon)/\epsilon}$

$$\mathbf{Prob} \left(\sum_i x_i r_i \geq \zeta \right) > 1 - \epsilon$$

0-1 problems

n	ϵ	% gap	nodes	time
25	.10	5.67	250	1
	.05	12.43	585	2
	.03	24.43	2345	6
	.02	32.62	843	3
	.01	43.21	315	1
50	.10	2.88	1129	34
	.05	6.46	4228	33
	.03	10.24	29957	214
	.02	14.67	98530	646
	.01	26.05	205290	1076
100	.10	0.96	3025	30
	.05	2.35	13375	145
	.03	4.96	76809	911
	.02	7.96	182603	$1873^\star$
	.01	15.81	204104	$1884^\star$

mixed 0-1 problems ($n = 100$)

m	ϵ	% gap	nodes	time
5	.10	0.81	556	7
	.05	2.09	10792	139
	.03	4.40	55452	788
	.02	6.97	149853	1859
	.01	13.68	167627	$1871^\star$
10	.10	0.74	569	8
	.05	1.91	9116	138
	.03	4.04	42451	709
	.02	6.38	99521	1511
	.01	12.15	139219	$1857^\star$
20	.10	0.63	571	10
	.05	1.63	8974	153
	.03	3.41	33259	665
	.02	5.29	72535	1386
	.01	9.97	114365	$1852^\star$

$$\min\left\{g(x) := ax + \Omega\sqrt{cx + \sigma^2} : x \in \{0,1\}^n\right\} \quad (\Omega, \sigma, c \geq \mathbf{0})$$

Greedy algorithm by Shen, Coullard, Daskin (2003)

Index variables s.t. $\frac{a_1}{c_1} \leq \cdots \leq \frac{a_n}{c_n}$. Let $S_i := \{1, 2, \ldots, i\}$ for $i = 1, 2, \ldots, n$.

Proposition

The set of all optimal solutions is some collection $\mathcal{S}$ of nested sets $S_{i_1} \subset S_{i_2} \subset \cdots \subset S_{i_k}$, $1 \leq k \leq n$.

Theorem

CLE_g is described completely by inequalities

$$\pi x \leq z - \sigma, \ \pi \in P_{g-\sigma}$$

and bound inequalities $0 \leq x \leq 1$.

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Given $\bar{x} \in \mathbb{R}^n_+$ and $\bar{z} \in \mathbb{R}$, is there a violated inequality

$$\pi x \leq z - \sigma, \quad \text{where} \quad \pi \in P_{g-\sigma} \ ?$$

$$\max \{\pi \bar{x} : \pi \in P_{g-\sigma}\} > \bar{z} - \sigma \ ?$$

Greedy algorithm by Edmonds (1971)

Given $\bar{x} \in \mathbb{R}^n_+$ and $\bar{z} \in \mathbb{R}$, is there a violated inequality

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Computations with 0-1 problems

n	ϵ	CPLEX			CPLEX + cuts			
		% gap	nodes	time	cuts	% gap	nodes	time
25	.10	5.67	250	1	4	0.73	47	0
	.05	12.43	585	2	4	1.06	115	1
	.03	24.43	2345	6	7	2.67	125	1
	.02	32.62	843	3	5	1.88	52	0
	.01	43.21	315	1	5	0.65	17	0
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	.02	7.96	182603	$1873^\star$	5	0.13	978	181
	.01	15.81	204104	$1884^\star$	10	0.26	2904	831

($\star$) Instances could not be solved in 30 mins.

Mixed 0-1 mean-risk minimization

$(\Omega, c, d \geq \mathbf{0})$

$$\min \left\{ h(x,y) := ax + by + \Omega \sqrt{cx + \sum_{i=1}^{m} d_i y_i^2} : (x,y) \in \{0,1\}^n \times [0,1]^m \right\}$$

h is concave in x, convex in y.

$$\mathcal{R}_h := \operatorname{conv} \left\{ (x,y,z) \in \{0,1\}^n \times [0,1]^m \times \mathbb{R} : h(x,y) \leq z \right\}$$

$$\pi x + by + \sqrt{\sum_{i \in T} d_i y_i^2} \leq z, \quad T \subseteq \{1,2,\ldots,m\} \qquad (CQ)$$

Proposition

CQ inequality is valid for $\mathcal{R}_h$ if and only if $\pi \in P_{f_T - \sigma_T}$.

$f_T(x) := h(x, \sum_{i \in T} e_i)$ and $\sigma_T := f_T(\mathbf{0})$, $T \subseteq \{1,2,\ldots,m\}$

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	.01	13.68	167627	$1871^\star$	7	2.12	107531	$1849^\star$
10	.10	0.74	569	8	2	0.19	124	2
	.05	1.91	9116	138	3	0.29	985	16
	.03	4.04	42451	709	4	0.67	12950	206
	.02	6.38	99521	1511	5	1.13	64403	1296
	.01	12.15	139219	$1857^\star$	6	2.86	80997	$1838^\star$
20	.10	0.63	571	10	3	0.09	109	2
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	.01	9.97	114365	$1852^\star$	8	3.52	40816	$1830^\star$

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Maximization of submodular utility

We consider the following mixed integer set

$$F := \left\{ \, x \in \{0,1\}^N, w \in \mathbb{R} : w \leq f(ax + d) \, \right\}$$

$a \in \mathbb{R}^N, \ d \in \mathbb{R}$ wlog $a > \mathbf{0}$

Assumption: $f : \mathbb{R} \to \mathbb{R}$ differentiable, strictly concave, increasing

Alternatively

$$F := \left\{ \, x \in \{0,1\}^N, w \in \mathbb{R} : ax \geq g(w) - d \, \right\}$$

$g := f^{-1}$ (differentiable, strictly convex, increasing)

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Maximizing expected utility in capital budgeting

N: set of candidate projects
r^i: return in scenario $i = 1, \ldots, m$
p_i: probability of scenario $i = 1, \ldots, m$

$$\max \left\{ \sum_{i=1}^{m} -p_i e^{-(r^i x)} : cx \leq b,\ x \in \{0,1\}^N \right\}$$

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$$\begin{array}{lll} & \max & w - cx \\ (REL) & \text{s.t.} & ax \geq g(w) - d \\ & & \mathbf{0} \leq x \leq \mathbf{1},\ w \in \mathbb{R} \end{array}$$

Proposition

There is a unique optimal solution (x, w) to (REL); moreover,

$$x_i = \begin{cases} 0, & \textit{if } c_i/a_i > 1/g'(w), \\ \frac{g(w)-a(T)-d}{a_i}, & \textit{if } c_i/a_i = 1/g'(w), \quad i \in N, \\ 1, & \textit{if } c_i/a_i < 1/g'(w), \end{cases}$$

where $T = \{k \in N : x_k = 1\}$.

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Special case: $f(ax) = -e^{-ax}$ (exponential utility)

$$(EXP) \qquad \max\left\{-ax - e^{ax} : x \in \{0,1\}^n\right\}$$

equivalently,

$$\max\left\{w - ax : ax \geq -\ln -w, x \in \{0,1\}^n, w \in \mathbb{R}\right\}$$

Proposition

Optimization problem (EXP) is $\mathcal{NP}$-hard.

$$F_1 = \{\ x \in \{0,1\}, w \in \mathbb{R} : w \leq f(ax)\ \}$$

$$x = 0 : \ w \leq f(0)$$

$$x = 1 : \ w \leq f(a)$$

$$w \leq f(0) + (f(a) - f(0))x$$

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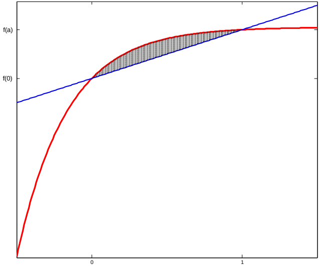

Let $h : 2^N \to \mathbb{R}$ s.t. $h(S) := f(a(S))$ for $S \subseteq N$.

PROPOSITION

(Nemhauser, Wolsey, Fisher 78) If h is a submodular function on N,

1. $h(T) \leq h(S) - \sum_{i \in S \setminus T} \rho_i(N \setminus i) + \sum_{i \in T \setminus S} \rho_i(S)$ *for all* $S, T \subseteq N$*;*
2. $h(T) \leq h(S) - \sum_{i \in S \setminus T} \rho_i(S \setminus i) + \sum_{i \in T \setminus S} \rho_i(\emptyset)$ *for all* $S, T \subseteq N$*.*

MIP formulation for submodular function maximization

$$w \leq h(S) - \sum_{i \in S} \rho_i(N \setminus i)(1 - x_i) + \sum_{i \in N \setminus S} \rho_i(S) x_i \quad \text{for all } S \subseteq N \quad (SM1)$$

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$$\max \left\{ \sum_{i=1}^{m} p_i w_i :\ cx \le b,\ w_i \le -e^{-(r^i x)}, i = 1, \dots, m,\ x \in \{0,1\}^N \right\}$$

n	m	Submodular inequality formulation cuts	gap(%)	nodes	time/(egap)
10	1	58	18.63	13	0
	25	1,572	22.27	31	0
	50	2,917	19.67	26	1
	100	4,241	15.10	22	1
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	25	88,263	98.39	10,890	(98.13)
	50	106,562	99.73	7,541	(99.70)
	100	124,804	99.76	4,804	(99.72)

30 of 60 instances not solved
27 instances hit 1GM memory limit
3 instances hit 1 hour time limit

Inequalities from $F(\emptyset, S)$

$$F = \left\{ x \in \{0,1\}^N, w \in \mathbb{R} : \sum_{i \in S} -a_i(1 - x_i) + \sum_{i \in N \setminus S} a_i x_i \geq g(w) - a(S) \right\}.$$

Fix $x_i = 1,\ i \in S$.

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Lift $w \le h(S) + \sum_{i \in N \setminus S} \rho_i(S) x_i$

$$
(L:S) \qquad
\begin{aligned}
\zeta(\delta) := \max\ & w - \sum_{i \in N \setminus S} \rho_i(S) x_i - h(S) \\
\text{s.t.}\ & \sum_{i \in N \setminus S} a_i x_i \ge g(w) - a(S) - \delta \quad (\delta \in \mathbb{R}_-) \\
& x \in \{0,1\}^{N \setminus S},\ w \in \mathbb{R}
\end{aligned}
$$

Maximization of a submodular function.

Proposition

Problem $(L:S)$ can be solved by the greedy algorithm that sets x_i, $i \in N \setminus S$ from zero to one in nonincreasing order of a_i until the sum exceeds $-\delta$.

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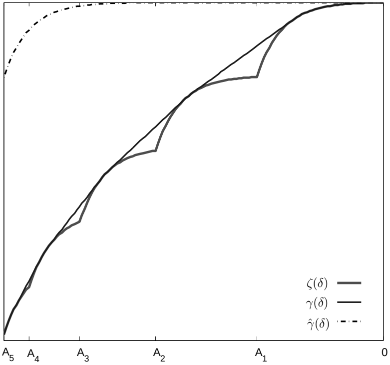
Assume $a_1 \geq a_2 \geq \cdots \geq a_{|N \setminus S|}$
Define $A_k = \sum_{i=1}^{k} a_i$ for $k = 1, \ldots, |N \setminus S|$, with $A_0 = 0$.

$$\zeta(\delta) = f(a(S) + A_k + \delta) - \sum_{i=1}^{k} \rho_i(S) - f(a(S)), \quad \text{for } -A_k \leq \delta \leq -A_{k-1}$$

ζ is continuous on $\mathbb{R}_-$ and piecewise concave.



The concave upper envelope of ζ over $\mathbb{R}_-$.

$$\gamma(\delta) = \begin{cases} \zeta(\mu_i - A_{i-1}) - \rho_i(S)\frac{b_i(\delta)}{a_i}, & \text{if } \mu_i - A_i \leq \delta \leq \mu_i - A_{i-1}, \\ \zeta(\delta), & \text{otherwise,} \end{cases}$$

where $\mu_i = g((g')^{-1}(a_i/\rho_i(S))) - a(S)$ and $b_i(\delta) = \mu_i - A_{i-1} - \delta$.

$$\zeta\Big(\sum_{i\in S} -a_i(1-x_i)\Big) \le \gamma\Big(\sum_{i\in S} -a_i(1-x_i)\Big) \le \sum_{i\in S}\gamma(-a_i)(1-x_i)$$

implying a subadditive lifting inequality

$$w \le \; h(S) + \sum_{i\in S}\gamma(-a_i)(1-x_i) + \sum_{i\in N\setminus S}\rho_i(S)x_i \qquad (L1)$$

valid for F.

Proposition

Inequality (L1) is facet-defining for $\mathrm{conv}(F)$ *if* $\gamma(a_i) = \zeta(a_i)$ *for all* $i \in S$*.*

Proposition

Inequalities (L1) cut off all fractional extreme points of $\mathrm{ContRel}(F)$*.*

Proposition

For each $S \subseteq N$ *inequality (L1) implies inequality (SM1).*

Inequalities from the restriction $F(N \setminus S, \emptyset)$

$$F = \left\{ x \in \{0,1\}^N, w \in \mathbb{R} : \sum_{i \in S} -a_i(1-x_i) + \sum_{i \in N \setminus S} a_i x_i \geq g(w) - a(S) \right\}.$$

Fix $x_i = 0, \ i \in N \setminus S$.

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$w \leq h(S) - \sum_{i \in S} \rho_i(S \setminus i)(1-x_i)$ is valid for $F(N \setminus S, \emptyset)$.

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Lift $w \leq h(S) - \sum_{i \in S} \rho_i(S \setminus i)(1 - x_i)$

$$
\begin{aligned}
\xi(\delta) := \max \ & w + \sum_{i \in S} \rho_i(S \setminus i)(1 - x_i) - h(S) \\
(L : N \setminus S) \qquad \text{s.t.} \ & \sum_{i \in S} -a_i(1 - x_i) \geq g(w) - a(S) - \delta \quad (\delta \in \mathbb{R}_+) \\
& x \in \{0,1\}^S, \ w \in \mathbb{R}
\end{aligned}
$$

Maximization of a submodular function

Proposition

Problem $(L : N \setminus S)$, it can also be solved by the greedy algorithm that sets binary variables from one to zero in nonincreasing order of a_i until the sum exceeds δ.

Lift $w \leq h(S) - \sum_{i \in S} \rho_i(S \setminus i)(1 - x_i)$

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$$\begin{aligned} & \xi(\delta) := \max\ w + \sum_{i \in S} \rho_i(S \setminus i)(1 - x_i) - h(S) \\ (L : N \setminus S) \qquad & \text{s.t.}\ \sum_{i \in S} -a_i(1 - x_i) \geq g(w) - a(S) - \delta \quad (\delta \in \mathbb{R}_+) \\ & x \in \{0,1\}^S,\ w \in \mathbb{R} \end{aligned}$$

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Assume $a_1 \geq a_2 \geq \cdots \geq a_{|S|}$.
Define $A_k = \sum_{i=1}^{k} a_i$ for $k = 1, \ldots, |S|$, with $A_0 = 0$.

$$\xi(\delta) = f(a(S) - A_k + \delta) + \sum_{i=1}^{k} \rho_i(S \setminus i) - f(a(S)),$$

for $A_{k-1} \leq \delta \leq A_k$ and $k = 1, \ldots, |S|$.

Concave upper envelope of ξ over $\mathbb{R}_+$:

$$\omega(\delta) = \begin{cases} \zeta(A_i - \mu_i) - \rho_i(S \setminus i)\frac{b_i(\delta)}{a_i}, & \text{if } A_{i-1} - \mu_i \leq \delta \leq A_i - \mu_i, \\ \xi(\delta), & \text{otherwise,} \end{cases}$$

where $\mu_i = a(S) - g((g')^{-1}(a_i/\rho_i(S \setminus i)))$ and $b_i(\delta) = A_i - \mu_i - \delta$.

Proposition

ω is the concave upper envelope of ξ and it is subadditive over $\mathbb{R}_+$.

$$\xi\Big(\sum_{i\in N\setminus S} a_i x_i\Big) \leq \omega\Big(\sum_{i\in N\setminus S} a_i x_i\Big) \leq \sum_{i\in N\setminus S} \omega(a_i) x_i$$

implying the subadditive lifting inequality

$$w \leq \ h(S) - \sum_{i\in S} \rho_i(S\setminus i)(1-x_i) + \sum_{i\in N\setminus S} \omega(a_i) x_i \quad (L2)$$

Proposition

Inequality (L2) is facet-defining for $\mathrm{conv}(F)$ *if* $\omega(a_i) = \xi(a_i),\ \forall i \in N \setminus S$.

Proposition

Inequalities (L2) cut off all fractional extreme points of $\mathrm{ContRel}(F)$.

Proposition

For each $S \subseteq N$ *inequality (L2) implies inequality (SM2).*

$$w \le h(S) - \sum_{i \in S} \rho_i(N \setminus i)(1 - x_i) + \sum_{i \in N \setminus S} \rho_i(S) x_i \quad (SM1)$$

$$w \le h(S) + \sum_{i \in S} \gamma(-a_i)(1 - x_i) + \sum_{i \in N \setminus S} \rho_i(S) x_i \quad (L1)$$

$$w \le h(S) - \sum_{i \in S} \rho_i(S \setminus i)(1 - x_i) + \sum_{i \in N \setminus S} \rho_i(\emptyset) x_i \quad (SM2)$$

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Given $(\bar{w}, \bar{x})$, separate with

$$w \le h(S) + \sum_{i \in N \setminus S} \rho_i(S) x_i \quad (1)$$

$$w \le h(S) - \sum_{i \in S} \rho_i(S \setminus i)(1 - x_i) \quad (2)$$

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Exponential utility

For exponential utility $f(ax) = -e^{-ax}$, given $(\bar{w}, \bar{x})$ inequality

$$\bar{w} \leq -e^{-a(S)} - \sum_{i \in N \setminus S} (e^{-a(S \cup i)} - e^{-a(S)})\bar{x}_i$$

is violated iff

$$0 < \max_{S \subseteq N} \ \bar{w} + e^{-a(S)} \left(1 + \sum_{i \in N \setminus S} (e^{-a_i} - 1)\bar{x}_i \right)$$

or

$$-1 < \max_{S \subseteq N} \bar{w} e^{a(S)} + \sum_{i \in N \setminus S} (e^{-a_i} - 1)\bar{x}_i$$

Let $z_i = 1$ if $i \in S$, $z_i = 0$ if $i \notin S$. Submodular maximization ($\bar{w} < 0$):

$$\underbrace{\frac{1}{\bar{w}} + \sum_{i \in N} (e^{-a_i} - 1)\frac{\bar{x}_i}{\bar{w}}}_{\text{constant}} + \max \sum_{i \in N} \frac{\bar{x}_i}{\bar{w}}(e^{-a_i} - 1) z_i + w$$

$$\text{s.t.} \sum_{i \in N} -a_i z_i \geq -\ln -w$$

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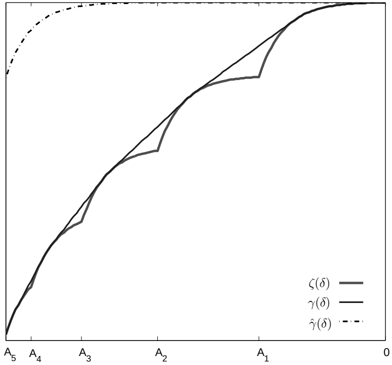
n	m	Submodular ineqs. cuts	gap(%)	nodes	time/(egap)	Lifted ineqs. cuts	gap(%)	nodes	time
10	1	58	18.63	13	0	18	9.10	11	0
	25	1,572	22.27	31	0	581	15.58	21	0
	50	2,917	19.67	26	1	1,051	11.55	19	0
	100	4,241	15.10	22	1	2,043	11.78	18	1
25	1	7,342	53.87	344	63	29	5.51	22	0
	25	57,308	73.90	3,559	(12.04)	4,580	9.54	215	9
	50	78,080	81.50	4,727	(43.45)	9,165	11.04	236	22
	100	113,466	83.38	2,942	(60.46)	17,608	9.61	196	44
50	1	66,140	98.16	60,448	(97.54)	162	3.56	91	0
	25	88,263	98.39	10,890	(98.13)	8,646	7.62	259	39
	50	106,562	99.73	7,541	(99.70)	13,830	8.57	315	58
	100	124,804	99.76	4,804	(99.72)	29,722	8.57	814	231



Inequalities SM1 & L1				Inequalities SM2 & L2			
$-\rho_i(N \setminus i)$	$\gamma(-a_i)$	$\zeta(-a_i)$	Imp(%)	$\rho_i(\emptyset)$	$\omega(a_i)$	$\xi(a_i)$	Imp(%)
-0.38	-382.78	-392.28	97.57	$1,268,076.31$	$3,232.78$	$3,161.37$	99.99

$(n = 50,\ m = 100)$

Nonlinear integer programs are hard!