Higher dimensional split closures

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Outline

◊ Lattice point free convex sets

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- Split bodies and cutting planes

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- ♢ Split bodies and cutting planes
- ♢ Structure of relaxations from split sets
- ♢ Width measures and split rank
- ♢ Higher dimensional split closures
- ♢ Polyhedrality of higher dimensional split closures

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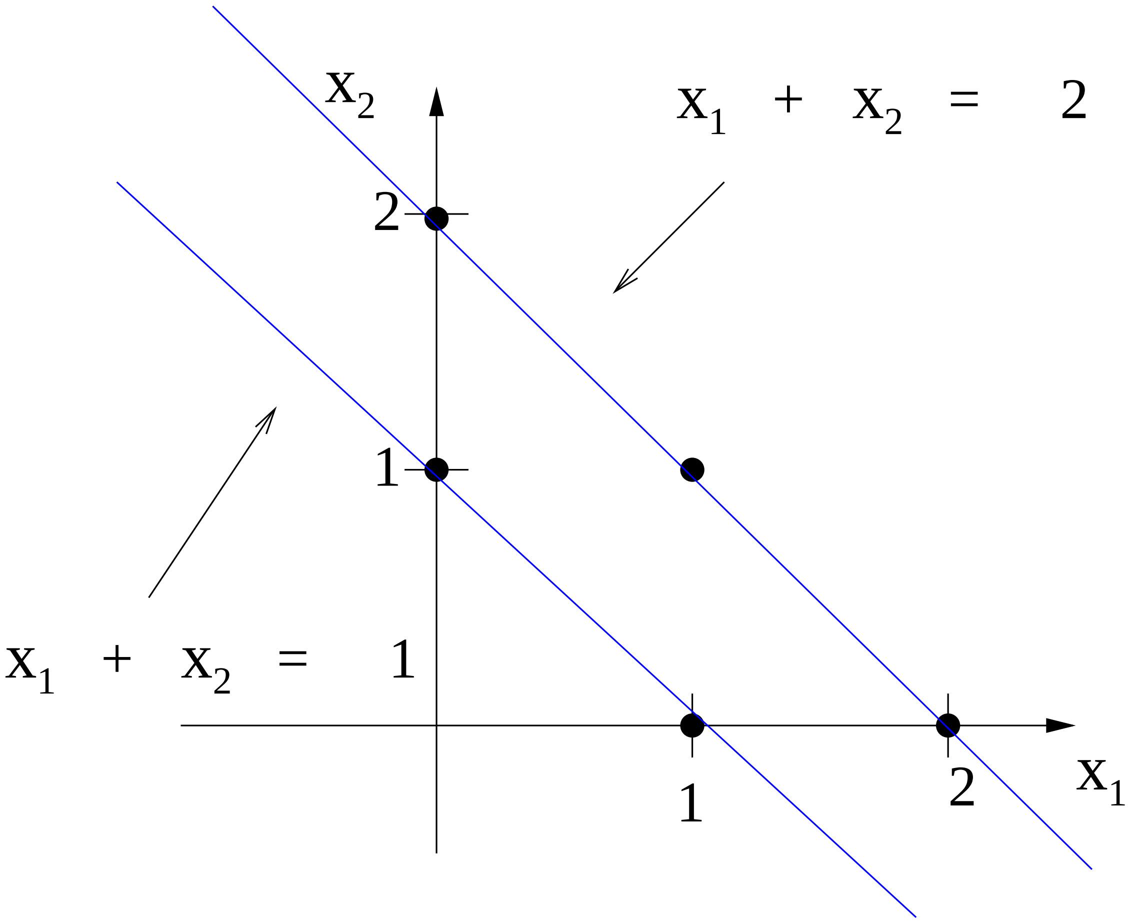
  - ⇒ Mixed integer rounding cuts (Nemhauser and Wolsey).

Split bodies : Examples



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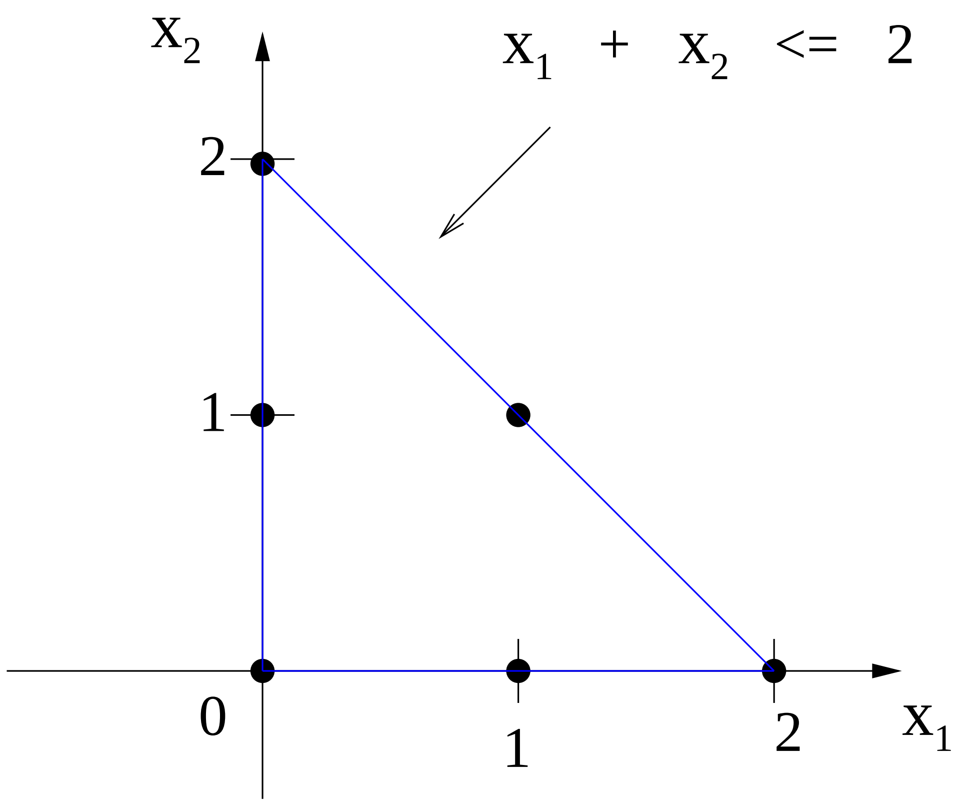

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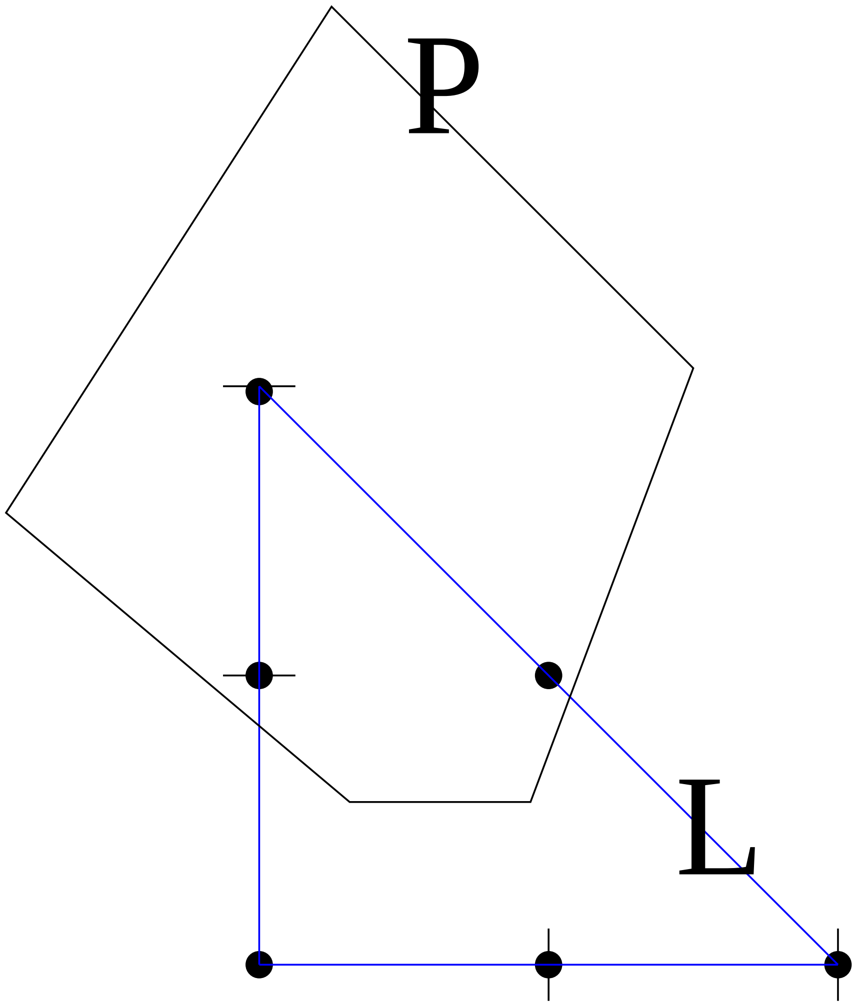
  - ⇒ $R(L, P)$ is a rational polyhedron.
  - ⇒ $R(L, P) \neq P$ iff $v \in \operatorname{int}(L)$ for some vertex v of P.
- ♢ We call valid inequalities for $R(L, P)$ higher rank split cuts.

A split body and an LP relaxation



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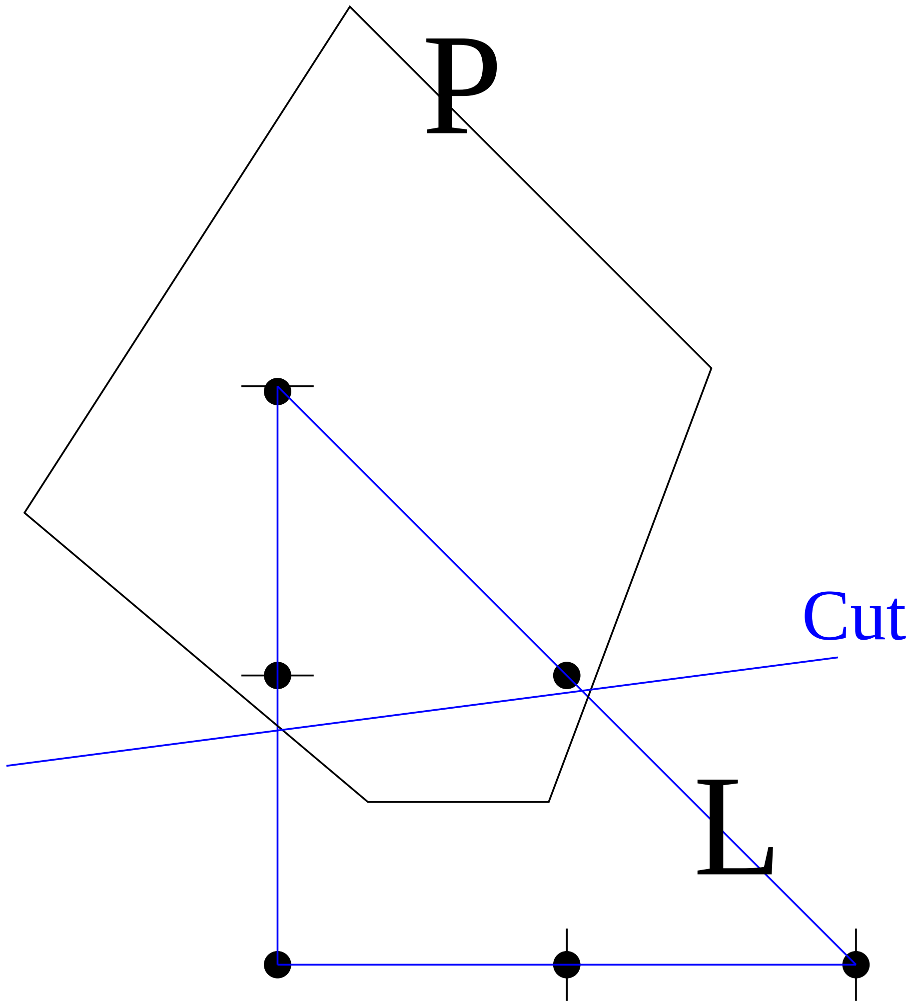

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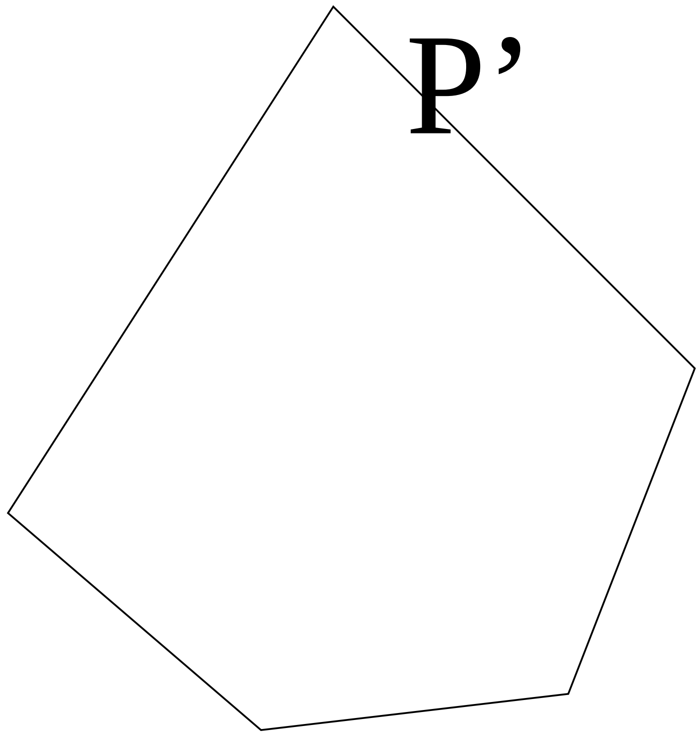

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- For $i \in V^{\text{in}}$ and $k \in V^{\text{out}}$, let $\beta_{i,k} \in]0, 1]$ be such that $\beta_{i,k} v^k + (1 - \beta_{i,k}) v^i$ is on the boundary of L.

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- We call $\beta_{i,k} v^k + (1 - \beta_{i,k}) v^i$ an intersection point

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  $x = (1 - \sum_{k \in V^{\text{out}}} \lambda_k) v^i + \sum_{k \in V^{\text{out}}} \lambda_k v^k$,
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- Hence P^i is the projection of the set:
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  onto the space of x-variables.
- Theorem:
  $R(L, Q^i) = \{(x, \lambda) \in Q^i : \sum_{k \in V^{\text{out}}} \frac{\lambda_k}{\beta_{i,k}} \geq 1\}$.

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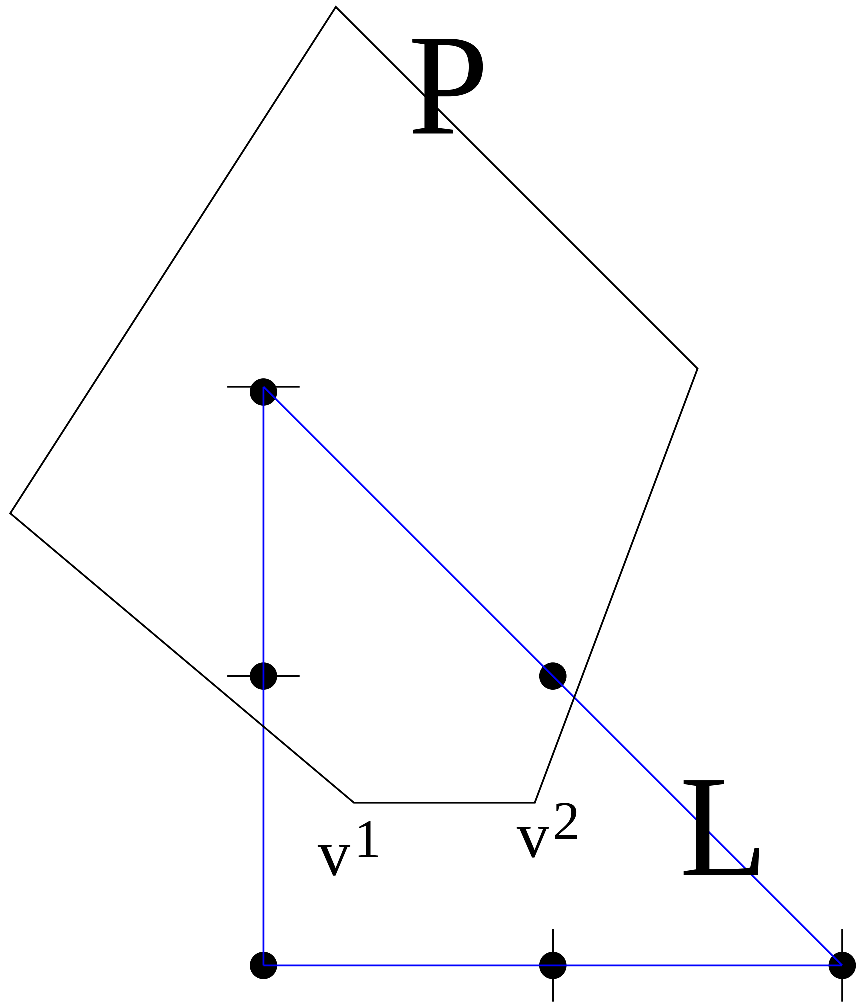
- $R(L, P)$ is completely characterized by the intersection points.

The example continued



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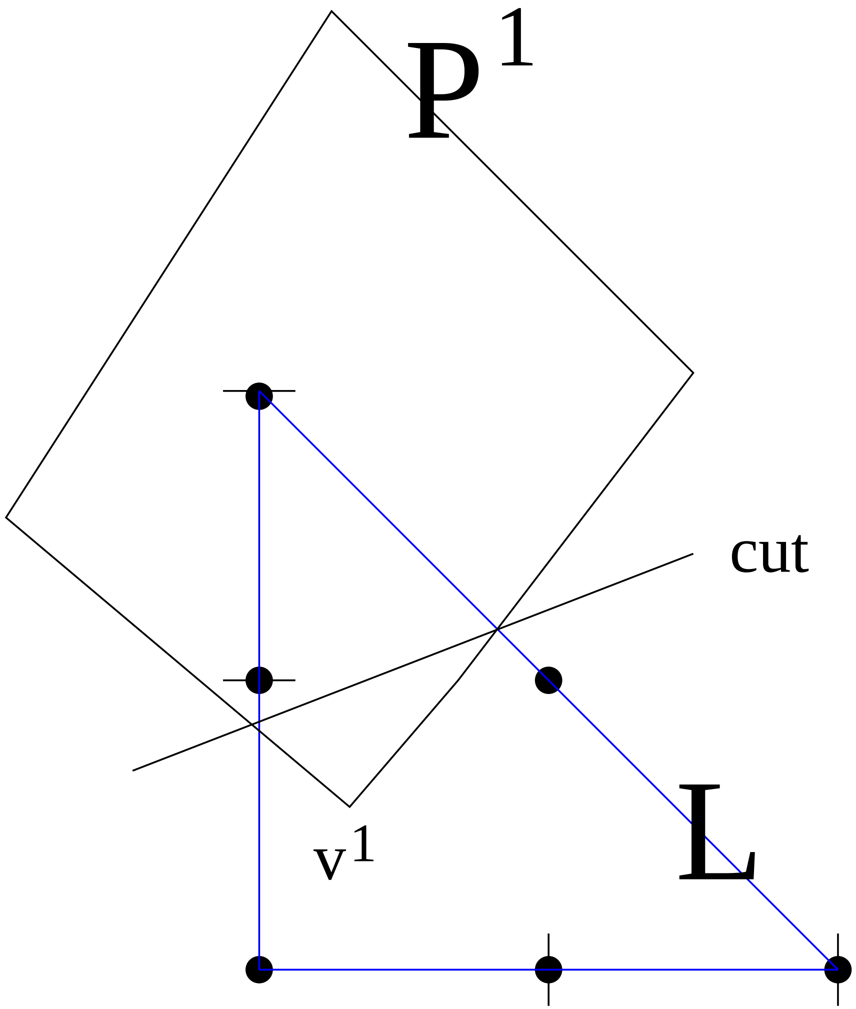

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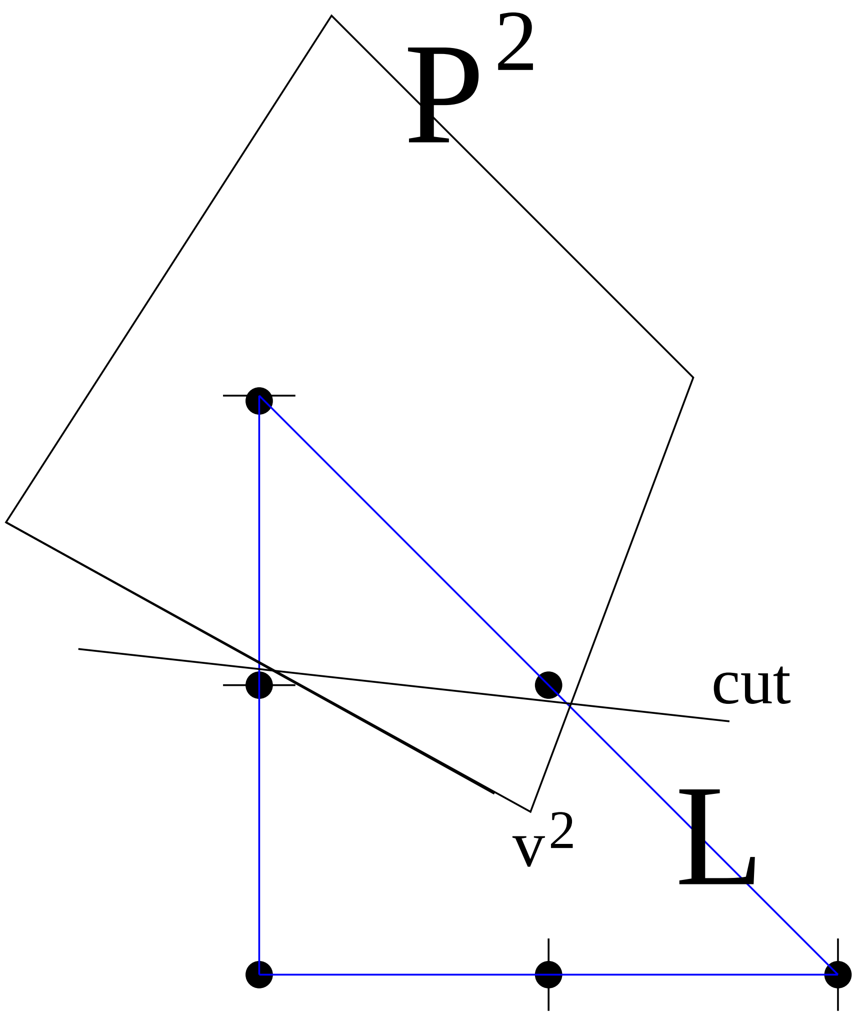

Width measures and split rank

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- Observe : any split set $\{x : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$ has max-facet-width equal to one.

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- Observe : any split set $\{x : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$ has max-facet-width equal to one.
- Our example : The set $\{x \in \mathbb{R}^2 : x \geq 0$ and $x_1 + x_2 \leq 2\}$ has max-facet-width equal to two.

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- The minimum $w \geq 1$ for which $\delta^T x \geq \delta_0$ is valid for $R(L, P)$ for some split body L with max-facet-width w is called the split rank of $\delta^T x \geq \delta_0$.

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- For a set $P_I \subseteq \mathbb{Z}^p \times \mathbb{R}^q$, the split rank of $\delta^T x \geq \delta_0$ depends on structure of the set:
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- Consider the mixed integer program:
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  Subject to
  $-x_i + y \leq 0$ for $i = 1, 2, \ldots, p$,
  $\sum_{i=1}^p x_i + y \leq p$,
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- The valid inequality $y \leq 0$ has split rank p.

Higher dimensional split closures

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of split bodies with max-facet-width at most w.

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- When $w = 1$, $\mathcal{L}^w$ consists of the (usual) split bodies $L_{\pi,\pi_0} = \{x : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$.

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- For any $w \geq 1$, the w^{th} split closure is defined to be:

  $\text{Cl}_w(P) := \cap_{L \in \mathcal{L}^w} R(L, P)$.

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- Our proof is based on a characterization of the facets of $R(L, P)$.

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- If we let $\bar{V} := V^c \cup (\cup_{i \in V^c} V^i)$ (all vertices above) and $P(\bar{V}) := \text{conv}(\{v^k\}_{k \in \bar{V}})$, then

$$R(L, P(\bar{V})) = \text{conv}(\cup_{i \in V^c} R(L, P^i(V^i))) = \{x \in P(\bar{V}) : \delta^T x \geq \delta_0\}.$$

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- ◇ We show: For a fixed B, only a finite number of non-dominated inequalities $(\delta(L, B))^T x \geq \delta_0(L, B)$ with $L \in \mathcal{L}^w(B)$ are needed.

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- We show: For a fixed B, only a finite number of non-dominated inequalities $(\delta(L, B))^T x \geq \delta_0(L, B)$ with $L \in \mathcal{L}^w(B)$ are needed.
- Since there is only a finite number of configurations B, this shows $\text{Cl}_w(P)$ is a polyhedron.