Dynamic Programming based Inequalities for the Capacitated Lot-sizing Problem

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Contribution

In our study, we aim to make contributions to the polyhedral characterization of the lot-sizing problems by defining a new set of valid inequalities for the capacitated lot-sizing problem(CLSP) that are derived from the end-of-stage solutions of a dynamic programming algorithm.

MIP Formulation for CLSP

$$\min \sum_{t=1}^{T} (p_t x_t + s_t y_t + h_t i_t) \qquad (1)$$

subject to:

$$i_{t-1} + x_t - d_t = i_t \qquad t = 1, \ldots, T \qquad (2)$$
$$x_t \le c_t y_t \qquad t = 1, \ldots, T \qquad (3)$$
$$i_t, x_t \ge 0 \qquad t = 1, \ldots, T \qquad (4)$$
$$y_t \in \{0, 1\} \qquad t = 1, \ldots, T \qquad (5)$$

Dynamic Programming Formulation

- We define the state of the system in period t as the inventory level at time t.
- The minimum inventory level at period $t = 1, \ldots, T$ in any feasible solution:
$$L_t = \max \left\{ 0, \max_{\tau = t+1, \ldots, T} \sum_{j=t+1}^{\tau} (d_j - c_j) \right\}$$
- The maximum inventory level at period $t = 1, \ldots, T$ in any optimal solution:
$$U_t = \min \left\{ C_t - D_t, \sum_{j=t+1}^{T} d_j \right\}$$
- Given inventory level i, $L_t \le i \le U_t$ for period t, production in any optimal solution at period t $X_{t,i}$ = {max{0, $i + d_t - U_{t-1}$},…,min{$c_t, i + d_t - L_{t-1}$}}.
- We define $F_t(i)$ as the minimum cost of solving the problem over the first t periods with an ending inventory level i.
- The recursion is given as
$$F_t(i) = \min_{x_t \in X_{t,i}} \{ p_t x_t + s_t y_t + h_t i + F_{t-1}(i + d_t - x_t) \},$$
$$\forall t = 1, \ldots, T, \; L_t \le i \le U_t,$$
$$F_0(0) = 0$$
- The optimal schedule is defined as $F_T(0)$.
- The complexity of the algorithm is $O(D_T^2)$.

DP Inequalities

Proposition: For any $t = 1, \ldots, T$, the following inequality is valid for CLSP:

$$\sum_{j=1}^{t} \left(p_j x_j + s_j y_j + h_j i_j \right) \ge F_t(L_t), \qquad (6)$$

since $F_t(i_1) < F_t(i_2)$ for $L_t \le i_1 < i_2 \le U_t$, and hence $F_t(L_t)$ represents the minimum cost of all feasible decisions through period t.

Stronger DP Inequalities

$$\sum_{j=1}^{t-1} \left(p_j x_j + s_j y_j + h_j i_j \right) + s_t y_t + p_t x_t \ge F_t(i) - h_t i_t \quad \text{if } i = i_t. \qquad (7)$$

Lemma: The following inequality is valid to CLSP:

$$\sum_{j=1}^{t-1} \left(p_j x_j + s_j y_j + h_j i_j \right) + s_t y_t + p_t x_t \ge F_t(L_t) - h_t L_t, \qquad (8)$$

and is at least as tight as (6).

Convex Envelope DP Inequalities

- We represent functional values as a function of the inventory at stage t.
- Then we take the convex envelope of this function.
- Convex Envelope DP inequalities have the form:
$$\sum_{j=1}^{t} \left(p_j x_j + s_j y_j + h_j i_j \right) \ge m_{tq} i + b_{tq} \qquad (9)$$
for parameters m_{tq} and b_{tq}, $q = 1, \ldots, Q_t$, where Q_t is the number of segments defining the convex envelope.

Tightness of the Convex Envelope Inequalities

- We now explore the tightness of the convex envelope inequality (9) and try to strengthen it by lifting where possible.
- We define $z_t = \sum_{j=1}^{t} (p_j x_j + s_j y_j + h_j i_j)$.
- For each stage $t = 1, \ldots, T$, we consider the polyhedron
$$P^t = \{ (\nu_t, y_{t+1}, \ldots, y_T) : \text{ there exists a feasible solution to the CLSP having values } (y_{t+1}, \ldots, y_T) \text{ and } \nu_t \ge z_t \},$$
i.e., P^t is the epigraph of the stage t value function, projected onto the dimensions $y_{t+1}, \ldots, y_T$.

Lemma: For $t \in \{1, \ldots, T\}$, $\dim(P^t) = T - t + 1$.

Proposition: For $t \in \{1, \ldots, T\}$, suppose that given an inventory level of i_{tq} for some $q \in \{1, \ldots, Q_t\}$, there does not exist a $u \in \{t+1, \ldots, T\}$ such that $y_u = 1$ in any feasible solution. Then (9) defines a facet to P^t.

Lifted Convex Envelope Inequalities

Given an inventory level i_{tq}, choose any u_1 such that y_{u_1} is fixed to 1 in any feasible solution, and find the largest value of α_{t1} for which:

$$z_t \ge m_{tq} i_t + b_{tq} + \alpha_{t1}(1 - y_{u_1}) \qquad (10)$$

is valid.
We wish to maximize α_{t1}, subject to:

$$\alpha_{t1} \le z_t - m_{tq} i_t - b_{tq}, \quad \forall (z_t, i_t) : \text{there exists a feasible solution having partial cost } z_t, \text{ period } t \text{ inventory } i_t, \text{ and } y_{u_1} = 0. \qquad (11)$$

Example Problem

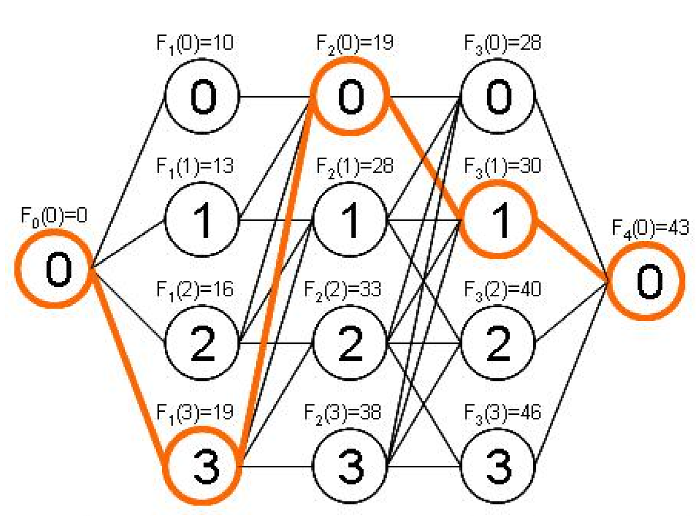

Figure: Graphical representation of an instance of CLSP with $T = 4$

Table: Data

t	c_t	d_t	p_t	s_t	h_t
1	5	2	1	8	2
2	3	3	2	7	2
3	4	3	1	6	1
4	3	3	3	7	1

DP Inequalities for the Example

The corresponding inequalities (6) and (8) for t=1 are

$$x_1 + 8y_1 + 2i_1 \ge 10 \qquad (12)$$
$$x_1 + 8y_1 \ge 10 \qquad (13)$$

and for t=2

$$x_1 + 8y_1 + 2i_1 + 2x_2 + 7y_2 + 2i_2 \ge 19 \qquad (14)$$
$$x_1 + 8y_1 + 2i_1 + 2x_2 + 7y_2 \ge 19 \qquad (15)$$

Convex Envelope Example

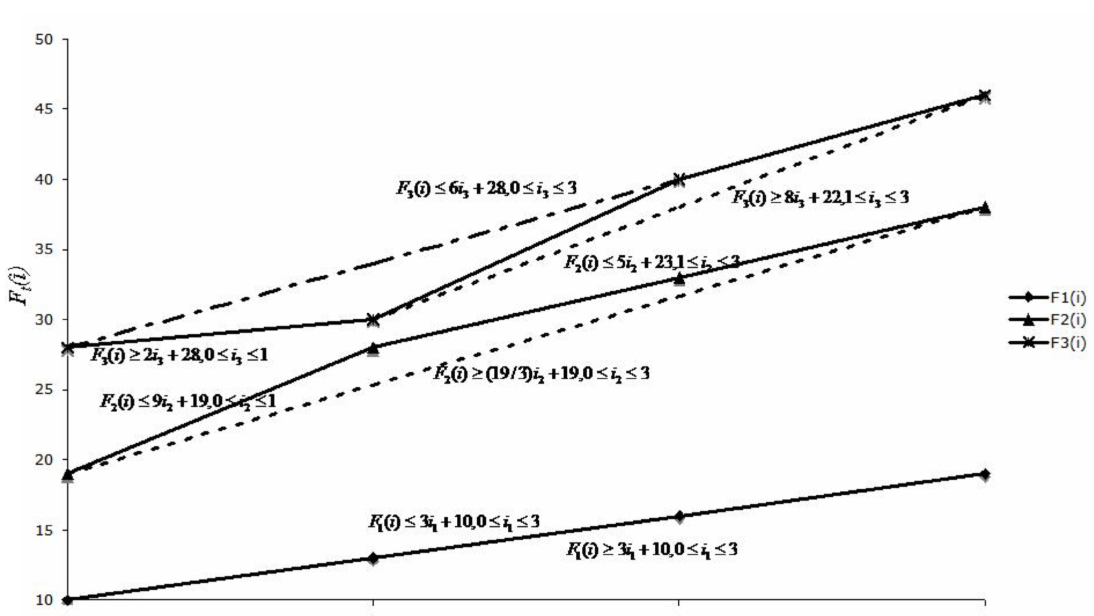

Figure: Graphical representation of $F_t(i)$ values and associated convex envelope inequalities.

Convex Envelope Inequalities

The convex envelope is defined by two inequalities for $t = 3$:

$$\sum_{j=1}^{3} \left(p_j x_j + s_j y_j + h_j i_j \right) \ge 2i_3 + 28, \qquad (16)$$
$$\sum_{j=1}^{3} \left(p_j x_j + s_j y_j + h_j i_j \right) \ge 8i_3 + 22. \qquad (17)$$

Lifted Convex Envelope Inequalities

Setting $u_1 = 4$, we can lift (16) by adding the term $\alpha_{t1}(1 - y_4)$ to the right-hand-side of (16),

$$\sum_{j=1}^{3} \left(p_j x_j + s_j y_j + h_j i_j \right) \ge 2i_3 + 28 + \alpha_{t1}(1 - y_4), \qquad (18)$$

and then by computing $\alpha_{t1} = 46 - 2(3) - 28$ as prescribed by (11). The resulting inequality is given by

$$z_3 \ge 2i_3 + 28 + 12(1 - y_4). \qquad (19)$$

Computational Results I

- CLSP instances are randomly generated with varying cost and capacity characteristics as presented in the paper of Atamtürk and Muñoz [1].
- We generated 240 instances with 90, 120 and 150 periods, 80 instances for each.

Table: Summary of experiments for $T = 90$, $T = 120$ and $T = 150$.

T	exp	stage	gap	gapimp	DPineq	nodes	time
90	base	-	6.22	-	-	50282	36
	weakl	30	4.74	23.30	30	20704	22
	envl	30	3.96	36.62	137	12803	19
	liftenvl	30	3.95	36.85	138	8399	11
	weakl+env	(30,30)	3.92	37.16	168	10836	20
	weakl+env	(75,30)	2.05	65.61	213	5077	18
120	base	-	6.12	-	-	889156	784
	weakl	80	2.82	52.22	80	81913	152
	envl	45	3.78	38.04	229	104181	143
	liftenvl	45	4.25	33.15	231	93960	135
	weakl+env	(30,30)	4.47	27.04	164	173197	264
	weakl+env	(100,45)	1.82	68.63	331	13564	56
150	base	-	5.56	-	-	1793270	1876
	weakl+env	(40,40)	3.84	30.87	236	578509	1053
	weakl+env	(120,45)	1.70	67.16	349	98367	361

Computational Results II

Table: Experiments for $T = 90$ and $f = 10000$.

	exp	stage	gap	gapimp	DPineq	DPineqtime	time
	base	-	6.82	-	0	0	1209
Instance 1	weakl+env	(10+10)	6.14	10.02	25	4	663
DP time = 179	weakl+env	(15+15)	5.57	18.32	42	10	154
	weakl+env	(30+30)	4.49	34.19	121	61	155
	base	-	9.02	-	0	0	368
Instance 2	weakl+env	(10+10)	7.09	21.31	31	8	304
DP time = 208	weakl+env	(15+15)	6.68	25.96	58	20	163
	weakl+env	(30+30)	5.47	39.33	149	100	236
	base	-	6.73	-	0	0	453
Instance 3	weakl+env	(10+10)	5.61	16.67	33	4	626
DP time = 212	weakl+env	(15+15)	4.98	25.92	51	10	79
	weakl+env	(30+30)	4.37	35.07	142	68	262
	base	-	7.32	-	0	0	40
Instance 4	weakl+env	(10+10)	5.96	18.57	35	5	73
DP time = 222	weakl+env	(15+15)	5.67	22.61	52	12	60
	weakl+env	(30+30)	4.88	33.39	123	70	153
	base	-	5.95	-	0	0	113
Instance 5	weakl+env	(10+10)	5.12	13.90	24	4	20
DP time = 216	weakl+env	(15+15)	4.92	17.24	38	10	44
	weakl+env	(30+30)	4.13	30.62	101	67	151
	base	-	7.17	-	0	0	437
Average	weakl+env	(10+10)	5.98	16.09	30	5	337
DP time = 207	weakl+env	(15+15)	5.56	22.01	48	12	100
	weakl+env	(30+30)	4.67	34.52	127	73	191

Conclusions and Future Directions

- We defined a new set of inequalities based on the stage information of DP formulation of the CSLP and used them to strengthen the equivalent MIP formulation.
- The computational experiments with the DP-based inequalities suggest that they are quite effective in solving lot-sizing problems when added to the problem formulation.
- We will explore the use of the DP based inequalities within problem domains that contain the CLSP constraints as a substructure, e.g. multi-item CLSP for which the use of DP by itself is not a computationally competitive algorithm.
- Additionally, we believe that a similar technique can be employed to other problems aside from lot-sizing for which both DP and MIP approaches exist. Our goal is to generalize this method as far as possible to maximize the breadth of problems that can benefit from our approach.

References

[1] Atamtürk A. and Muñoz J. C., A Study of the Lot-sizing Polytope, *Mathematical Programming*, 99(3): 443-465, 2004.
[2] Hartman, J. C., Büyüktahtakın, İ. E. and Smith, J. C, Dynamic programming based inequalities for the capacitated lot-sizing problem, *Research Report #2008-2*, Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611.