Optimizing over the MIR closure of polyhedra

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Cutting planes

LP Relaxation:

$$\min c^T x$$

$$Ax \geq b$$

Tighten:

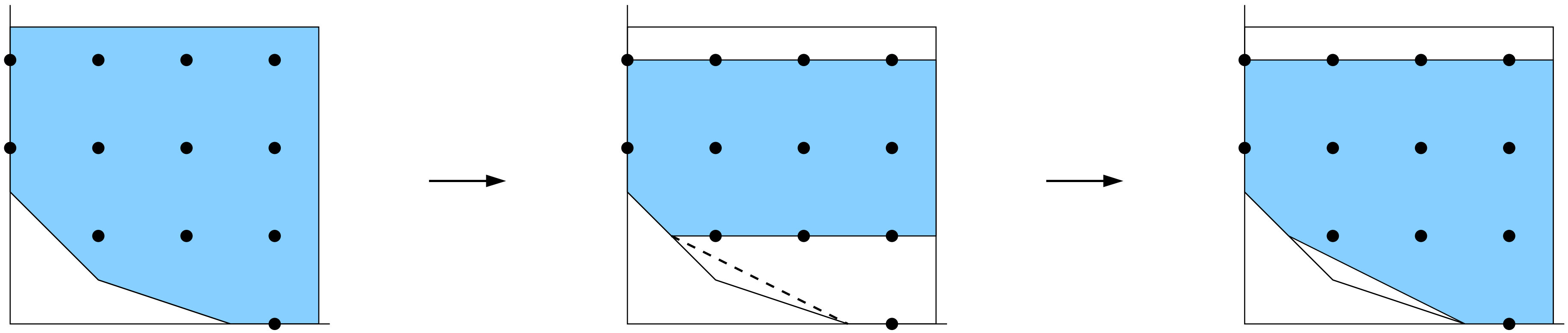
$$\min c^T x$$

$$Ax \geq b$$

$$c_1^T x \geq d_1$$

$$c_2^T x \geq d_2$$

$$\vdots$$

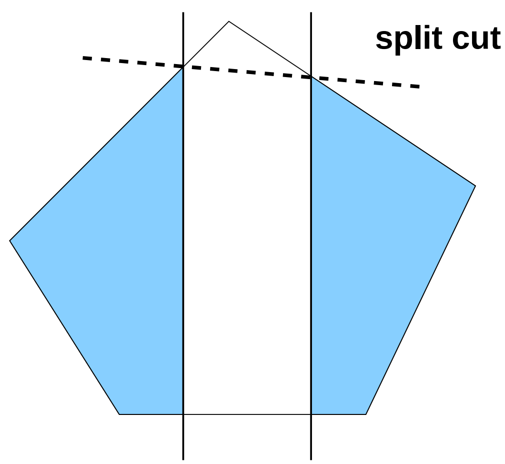


Split cuts

Idea: x integer $\Rightarrow x \leq 1$ or $x \geq 2$.

All integer solutions in P satisfy either $\bar{a}^T x \leq \bar{b}$ or $\bar{a}^T x \geq \bar{b} + 1$ where $\bar{a}$ and $\bar{b}$ are integral. A **split cut** is an inequality valid for:

$$P \cap \{\bar{a}^T x \leq \bar{b}\} \text{ and } P \cap \{\bar{a}^T x \geq \bar{b} + 1\}$$



Split cuts $\equiv$ MIR cuts $\equiv$ GMI cuts (Nemhauser, Wolsey '90).

Properties

Split closure: Set of points satisfying all split cuts.

Thm: Split closure of P is a polyhedron (Cook, Kannan, Schrijver '93).

Thm: It is NP-hard to find violated split cuts (Caprara, Letchford '03).

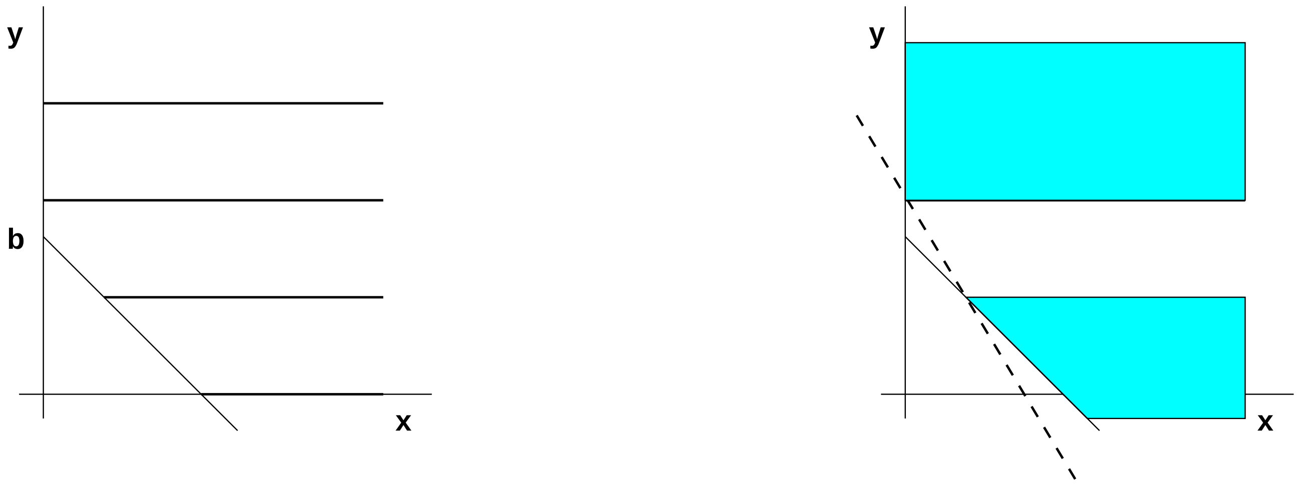
Thm: Every 0-1 split cut proof of *clique vs. coloring* has exponential length (Dash '05).

Q: Can the problem of finding violated split cuts be modeled as an MIP ? (Can be modeled as a non-convex QP.)

Basic mixed-integer inequality

$$Q^1: \quad x + y \ \geq \ b,$$

$x \geq 0, \quad y$ integral.



MI(R) Inequality:

$$\frac{x}{\hat{b}} + y \geq \lceil b \rceil \, .$$

(Wolsey '96, Marchand, Wolsey '01, Nemhauser, Wolsey '88)

MIR cuts

$$P : Ax + Cv = d$$
$$x \geq 0,\ v \geq 0$$

Valid inequalities:

$$a^T x + c^T v \geq b \text{ is valid for } P \Leftrightarrow \begin{array}{l} a^T \geq \lambda^T A,\ \ b \leq \lambda^T d \\ c^T \geq \lambda^T C \end{array}$$

MIR cuts for P:

Let $\hat{a} + \bar{a} \geq a,\qquad \bar{a}_i \in Z,\ \hat{a}_i \in [0,1)$

$c_+ \geq c,\qquad c_+ \geq 0$

Then $c_+^T v + (\hat{a} + \bar{a})^T x \geq b$ is valid for P,

and $(c_+^T v + \hat{a}^T x) + \hat{b}\bar{a}^T x \geq \hat{b}\lceil b \rceil$ is an MIR cut for P.

MIR cuts

MIR separation problem (MIR-sep):

Given $(v^*, x^*) \in P$, find MIR cut for P violated by (v^*, x^*).

Equivalent to:

$$\min c_+^T v^* + \hat{a}^T x^* + \hat{b}\bar{a}^T x^* - \hat{b}\lceil b \rceil$$

$s.t.\ c_+,\ \hat{a},\ \bar{a},\ b$ define an MIR cut.

Q1: Can MIR-sep be modeled as a (practical) MIP ?

Q2: How useful is the split closure for practical MIPs ?

Fischetti and Lodi ’05: Studied Q1 and Q2 for the Chvatal closure
Balas and Saxena ’06: MIR closure, Vielma ’05

Approximate MIR-sep model

$$P : Ax + Cv = d$$
$$x \geq 0,\ v \geq 0$$

MIR-sep:

$$\begin{aligned} \min \quad & c_+^T v^* + \hat{a}^T x^* - & \hat{b}(\lceil b \rceil - \bar{a}^T x^*) \\ s.t. \quad & \hat{a} + \bar{a} \geq \lambda^T A, & \bar{a}_i \in Z, \hat{a}_i \in [0,1) \\ & c_+ \geq \lambda^T C, & c_+ \geq 0 \\ & b \leq \lambda^T d. & \end{aligned}$$

Main Ideas:

1. Approximate $\hat{b}$ as $\hat{b} = \sum_{i=1}^{k} \epsilon_i \pi_i$ where $\epsilon_i = \frac{1}{2^i},\ \pi_i \in \{0,1\}$.

2. For any violated MIR cut, if $\Delta = \lceil b \rceil - \bar{a}^T x^*,$ then $0 < \Delta < 1$.

Approximate MIR-sep model

$$\begin{aligned} \min \quad & c_+^T v^* + \hat{a}^T x^* - \sum_{i=1}^{k} \epsilon_i \Delta_i \\ s.t. \quad & \text{MIR conditions} \quad + \\ & \Delta = \lceil b \rceil - \bar{a}^T x^*, \\ & \Delta_i \leq \pi_i \\ & \Delta_i \leq \Delta \\ & \Delta_i \geq 0 \\ & \pi_i \in \{0, 1\}. \end{aligned}$$

Idea: $\hat{b}\Delta \approx (\sum_{i=1}^{k} \epsilon_i \pi_i)\Delta = \sum_{i=1}^{k} \epsilon_i \Delta_i$

Exact MIR-sep model

Lemma: The multipliers λ_i can be assumed to lie in $(-m\delta, m\delta)$ in an optimal solution to MIR-sep, where m is the number of rows in P, and δ is the maximum value of sub-determinants of C. If $C = 0, \delta$ can be assumed to be $1/m$.

Cor 1: The MIR closure of P is a polyhedron.

Cor 2: $\hat{b}$ can be assumed to have finite precision.

Cor 3: MIR-sep can be solved as an MIP.

Proof

Let $\quad (\lambda C)^+ v + \hat{a}^T x + \hat{b}\bar{a}^T x \quad \geq \hat{b}\lceil b \rceil$ be violated by (v^*, x^*)

$i.e. \quad (\lambda C)^+ v^* + \hat{a}^T x^* + \hat{b}\bar{a}^T x^* \quad < \hat{b}\lceil b \rceil.$

Recall $\quad \hat{a} + \bar{a} \geq \lambda A, \qquad \bar{a}_i \in Z, \hat{a}_i \in [0,1)$

$$\hat{b} + \lfloor b \rfloor \leq \lambda d, \quad \hat{b} \in [0,1)$$

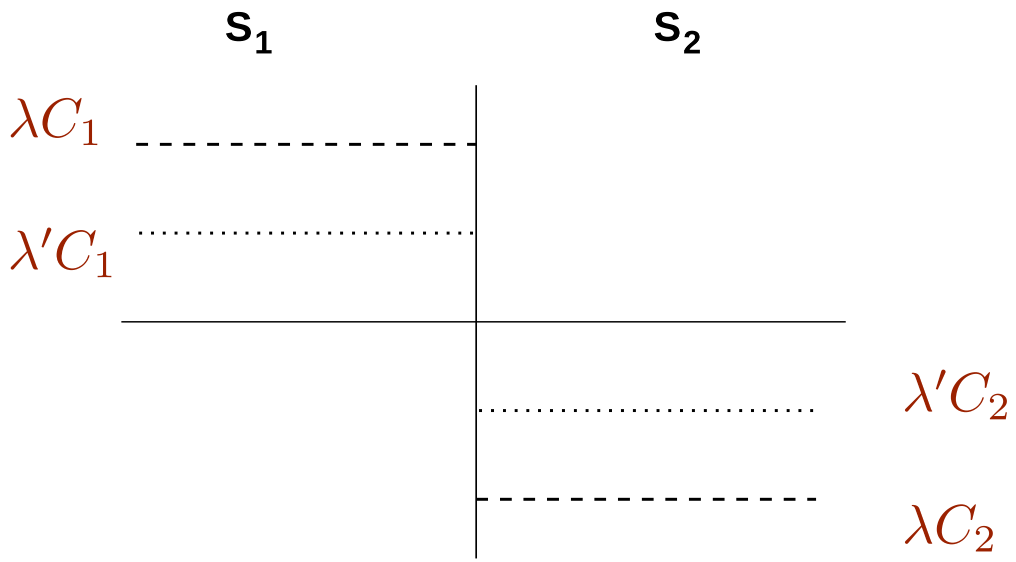
Define $\quad S_1 = \{i : \lambda C_i > 0\}, \;\; S_2 = \{i : \lambda C_i \leq 0\},$

$$C = \{v : vC_i \geq 0 \;\forall i \in S_1, \;\; vC_i \leq 0 \;\forall i \in S_2\}$$

Thus $\quad \lambda = \sum_i t_i \mu_i$ where μ_i are generators of C.

Define $\quad \lambda' = \sum_i \hat{t}_i \mu_i \Rightarrow \lambda - \lambda' = \bar{\lambda} = \sum_i \lfloor t_i \rfloor \mu_i.$

Claim: The violation of the MIR cut defined by λ' is at least as much as the violation of the MIR cut defined by λ.



Fact: $(\lambda C)^+ - (\lambda' C)^+ = (\bar{\lambda} C)^+$.

The rhs of the old MIR minus the rhs of the new MIR equals

$$\begin{aligned}\hat{b}\bar{\lambda}d = \hat{b}\bar{\lambda}(Ax^* + Cv^*) &= \hat{b}\bar{\lambda}Ax^* + \hat{b}\bar{\lambda}Cv^* \\ &\leq \hat{b}\bar{\lambda}Ax^* + \hat{b}(\bar{\lambda}C)^+v^*.\end{aligned}$$

Practical issues

1.**Number of basic variables**
Ignore variables at bounds; e.g., **nw04** has 36 constraints, 87000+ 0-1 vars, but separation MIP has 36+ integer vars.

2. **Cut selection:** objective

Obj 1 $\quad (\lambda C)^+ v^* + \hat{\alpha} x^* - \hat{\beta}\Delta$

Obj 2 $\quad \dfrac{(\lambda C)^+ v^* + \hat{\alpha} x^*}{\hat{\beta}} - \Delta$

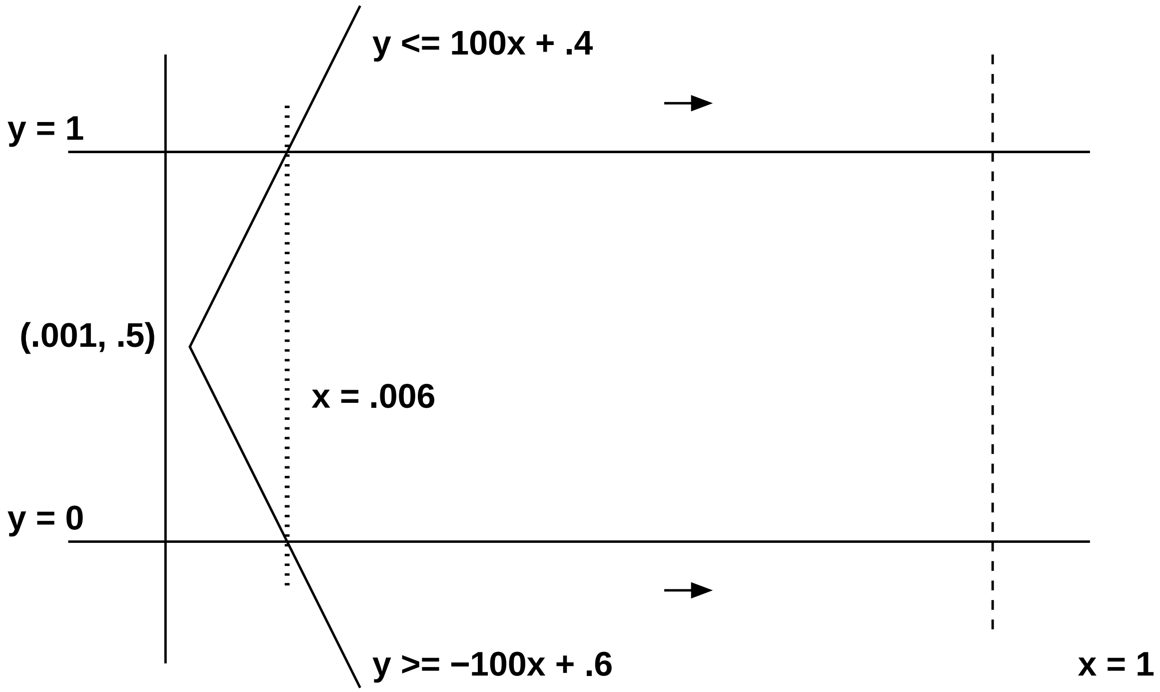
3. **Cut selection:** scaling

base: $\quad x \geq .5 \quad \Leftrightarrow \quad 101x \geq 50.5$

cut: $\quad x \geq 1 \quad \not\Leftrightarrow \quad 101x \geq 51$

4. **Precision**
LP solvers return approximate solutions of formulation + added cuts

Example



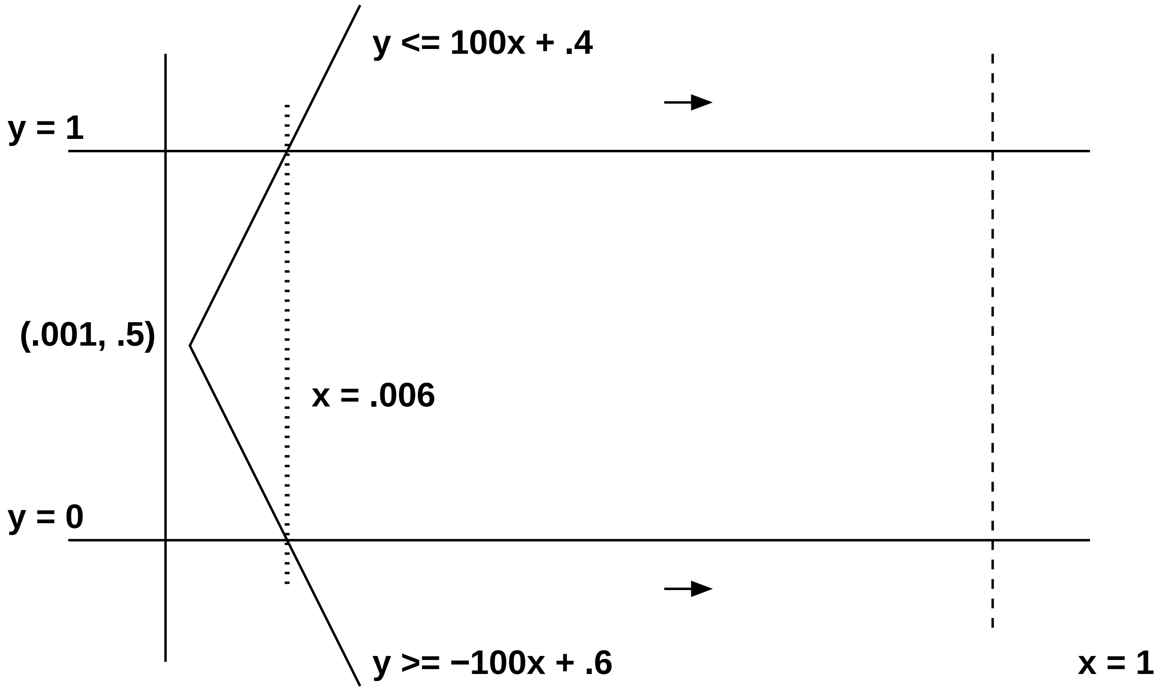
$$
\begin{aligned}
-100x + y + s_1 &= 0.4 \\
100x + y - s_2 &= 0.6
\end{aligned}
$$

$x \geq .001 \quad \rightarrow \quad x \geq 1 \qquad$ Obj 1 $= -\hat{b}\Delta \approx -.001$

$.5s_1 + y \geq .5 \quad \rightarrow \quad s_1 + y \geq 1 \qquad$ Obj 1 $= -\hat{b}\Delta = -.25$

$x \geq .006$

Example



$$\begin{aligned} -100x + y + s_1 &= 0.4 \\ 100x + y - s_2 &= 0.6 \end{aligned}$$

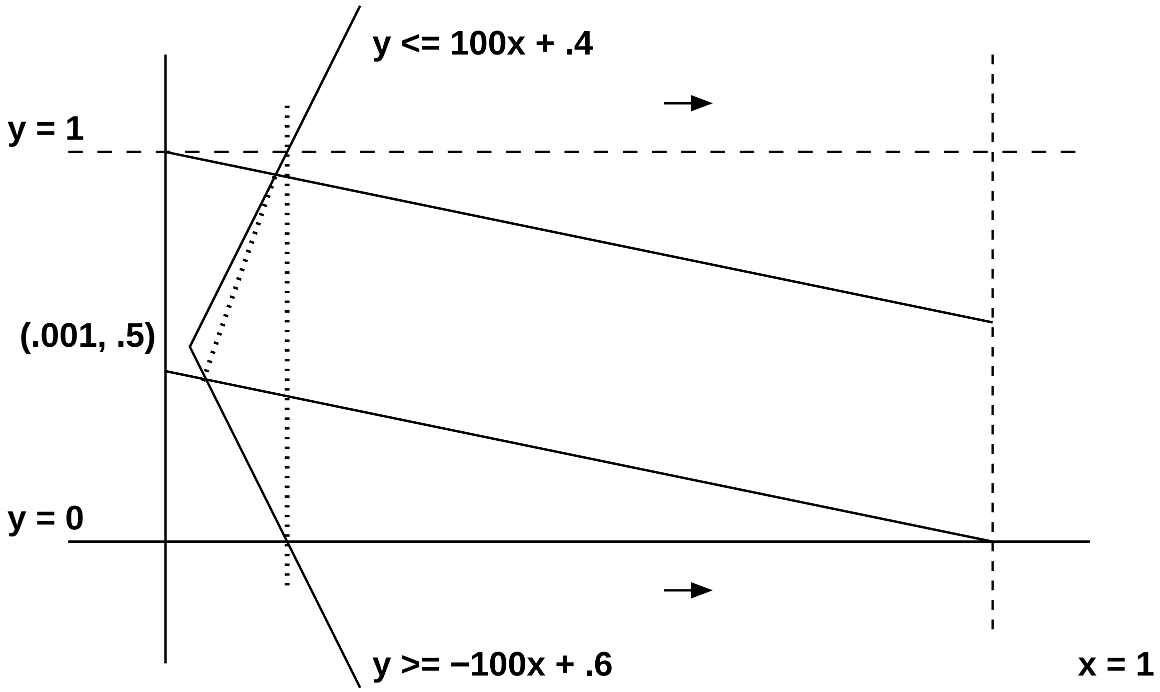
$x \geq .001 \quad \rightarrow \quad x \geq 1$ Obj 1 $= -\hat{b}\Delta \approx -.001$

Obj 2 $= -\Delta = -.999$

$.5s_1 + y \geq .5 \quad \rightarrow \quad s_1 + y \geq 1$ Obj 1 $= -\hat{b}\Delta = -.25$

$x \geq .006$ Obj 2 $= -\Delta = -.5$

Example



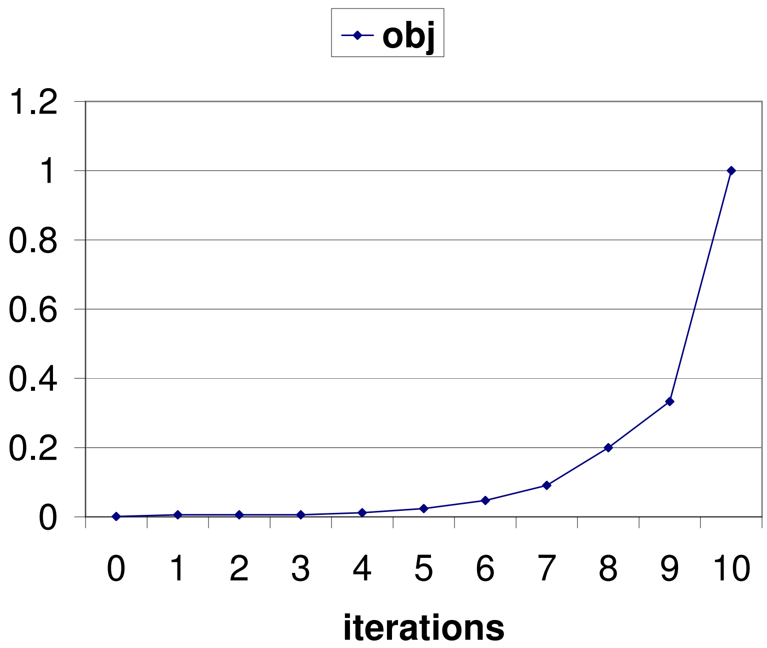
$$
\begin{aligned}
-100x + y + s_1 &= 0.4 \\
100x + y - s_2 &= 0.6
\end{aligned}
$$

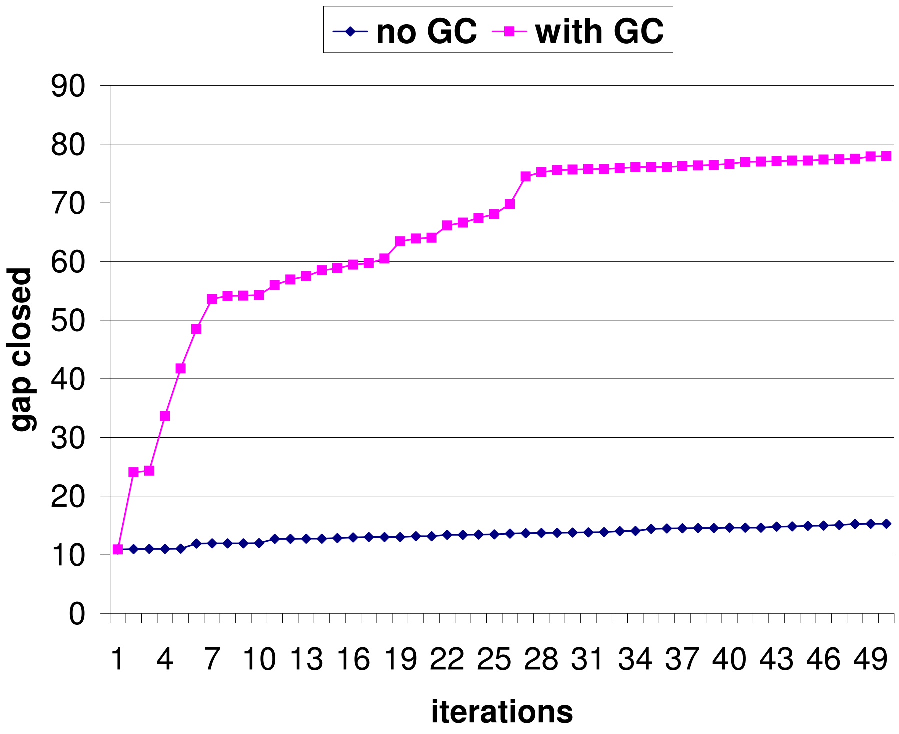
$x \geq .001 \quad \rightarrow \quad x \geq 1 \quad$ Obj 2 $= -\Delta = -.999$

$s_1 + x + 2y \geq 1.001 \quad \rightarrow \quad 1000s_1 + x + 2y \geq 2 \quad$ Obj 2 $= -\Delta = -.999$

Combined models

Use both GC separation and MIR separation: improvement on fixnet6





Separation Heuristic

- Add GMI cuts based on optimal tableau of LP relaxation

- Repeat

    Generate MIR cuts from formulation rows

    Generate violated G-C (pro-GC) cuts

    Generate violated MIR cuts

Instance	GMI	(pro)GC	MIR	BS
10teams	100.00	57.14	100.0	=
bell3a	45.10	48.10	99.63	
bell5	14.53	91.73	93.11	+
cap6000	41.65	26.90	52.62	
dcmulti	47.65	47.25	98.53	=
egout	55.63	81.77	100.0	=
flugpl	11.74	19.19	100.0	=
fixnet6	12.88	67.51	97.44	=
harp2	24.07	29.00	55.95	+
khb05250	74.91	4.77	100.0	=
misc06	28.47	0.0	100.0	=
rgn	3.15	0.0	99.63	=
p0033	56.82	85.40	87.42	=
p0201	26.71	60.50	74.28	=
pp08a	54.42	4.32	95.20	=
pp08aCUTS	33.79	0.68	92.22	=
vpm2	10.93	62.86	80.60	-

More Numbers ..

Instance	GMI	(pro)GC	MIR	BS
gesa2	28.53	94.84	95.89	=
gesa3	50.54	64.53	84.69	-
gesa3_o	47.53	58.96	71.32	-
modglob	17.28	0.0	80.91	
p2756	0.54	69.20	81.20	*
qnet1_o	42.99	8.61	82.80	-
rentacar	28.07		26.18	
set1ch	39.16	51.41	76.94	-
swath	8.42	7.68	33.19	

More Numbers ..

Instance	GMI	(pro)GC	MIR	BS
air04	6.95	27.60	8.22	
air05	4.64	15.50	6.54	-
arki001	29.26	28.04	34.86	
blend2	15.98	36.40	36.23	-
dano3mip	0.11	0.0		-
danoint	1.73	7.44		-
l152lav	10.88	69.20	12.13	-
mod011		0.00	17.21	
misc03	7.24	51.20	44.83	*
misc07	0.00	16.10	11.21	*
qiu	2.53	10.71	21.36	-
qnet1	11.91	7.32	70.48	-
rout	0.81	0.03	18.24	-
seymour	7.69	23.50	8.35	-