A Constructive Characterization of the Split Closure of a Mixed Integer Linear Program

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Review: MIP and Relaxation

We study the MIP feasible region

$$P_I := \{x \in P \subseteq \mathbb{R}^n : x_j \in \mathbb{Z} \quad \forall\, j \in N_I\}$$

where $N = \{1, \ldots, n\}$, $N_I \subseteq N$ and

$$P := \{x \in \mathbb{R}^n : Ax \le b\} \neq \emptyset$$

where $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $M = \{1, \ldots, m\}$.

Let

$$\mathcal{B}_r^* := \{B \subseteq M \;:\; |B| = r \text{ and } \{a_{i\cdot}\}_{i\in B} \text{ are linearly independent}\}.$$

where $r = \text{rank}(A)$ and $a_{i\cdot}$ corresponds to row i of A. For $B \in \mathcal{B}_r^*$ let $\bar{B}$ be the sub-matrix of A induced by B and $\bar{b}$ the sub-vector of b induced by B.

For $B \in \mathcal{B}_r^*$ let

$$P(B) := \{x \in \mathbb{R}^n \;:\; \bar{B}x \le \bar{b} \quad \forall\, i \in B\} \subseteq P.$$

and $x(B)$ a particular, but arbitrarily selected, solution to $\bar{B}x = \bar{b}$.

Review: Valid Split Disjunctions for MIP

For $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ we have the split disjunction

$$D(\pi, \pi_0) := \pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$$

and associated feasible region

$$F_{D(\pi,\pi_0)} := \{x \in \mathbb{R}^n \,:\, \pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1\}$$

We are interested in $D(\pi, \pi_0)$ such that

$$P_I \subseteq F_{D(\pi,\pi_0)} \subsetneq \mathbb{R}^n$$

so we study

$$\Pi_0^n(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} \,:\, \pi_j = 0,\, j \notin N_I\}$$

and its projection into the π variables

$$\Pi^n(N_I) := \{\pi \in \mathbb{Z}^n \setminus \{0\} \,:\, \pi_j = 0,\, j \notin N_I\}.$$

Review: Split Closure

The *split closure* [6] of P_I is

$$SC := \bigcap_{(\pi,\pi_0)\in \Pi_0^n(N_I)} \text{conv}(P \cap F_{D(\pi,\pi_0)}).$$

Theorem 1. *[6] SC is a polyhedron*

For $B \in \mathcal{B}_k^*$ let

$$SC(B) := \bigcap_{(\pi,\pi_0)\in \Pi_0^n(N_I)} \text{conv}(P(B) \cap F_{D(\pi,\pi_0)}).$$

Theorem 2. *[1]* $SC = \bigcap_{B\in\mathcal{B}_r^*} SC(B)$

Theorem 3. *[1] $SC(B)$ is a polyhedron for all $B \in \mathcal{B}_k^*$.*

Corollary 1. *[1] SC is a polyhedron*

Neither [1] nor [6] give constructive proofs.

Review: Characterization of Split Cuts

Proposition 1. *[1,3,5] All non-dominated valid inequalities for* conv$(P \cap F_{D(\pi,\pi_0)})$ *are of the form* $\delta(\mu,\pi,\pi_0)^T x \leq \delta_0(\mu,\pi,\pi_0)$ *where*

$$\delta(\mu,\pi,\pi_0) := \mu_0^1 \pi + \sum_{i\in M} \mu_i^1 a_{i\cdot} \;= -\mu_0^2 \pi + \sum_{i\in M} \mu_i^2 a_{i\cdot}$$

$$\delta_0(\mu,\pi,\pi_0) := \mu_0^1 \pi_0 + \sum_{i\in M} \mu_i^1 b_i \;= -\mu_0^2(\pi_0+1) + \sum_{i\in M} \mu_i^2 b_i$$

for $\mu_0^1, \mu_0^2 \in \mathbb{R}_+$ *and* $\mu^1, \mu^2 \in \mathbb{R}_+^m$ *solutions to*

$$\sum_{i\in M} \mu_i^2 a_{i\cdot} - \sum_{i\in M} \mu_i^1 a_{i\cdot} = \pi \qquad (1)$$

$$\sum_{i\in M} \mu_i^2 b_i - \sum_{i\in M} \mu_i^1 b_i - \mu_0^2 = \pi_0 \qquad (2)$$

$$\mu_0^1 + \mu_0^2 = 1 \qquad (3)$$

$$\mu_0^2 \in (0,1) \qquad (4)$$

$$\mu_i^1 \cdot \mu_i^2 = 0 \qquad \forall i \in M. \qquad (5)$$

Applying Proposition 1 to $P(B)$

Proposition 2. *For any $B \in \mathcal{B}_r^*$ if*

$$\begin{aligned} \bar{B}^T\mu &= \pi & \mu &\in \mathbb{R}^r \\ \mu^T\bar{b} &\notin \mathbb{Z} & \pi_0 &= \lfloor \mu^T\bar{b} \rfloor \end{aligned} \tag{6}$$

has no solution then $\text{conv}(P(B) \cap F_{D(\pi,\pi_0)}) = P(B)$*.* *If* (6) *has a (unique) solution $\bar{\mu}$ then*

$$\text{conv}(P(B) \cap F_{D(\pi,\pi_0)}) = \{x \in P(B) \,:\, \delta(\bar{\mu}, B)x \leq \delta_0(\bar{\mu}, B)\} \subsetneq P(B).$$

where $\delta(\bar{\mu}, B)x \leq \delta_0(\bar{\mu}, B)$ is defined in any of the following equivalent ways

$$(\bar{\mu}^-)^T(\bar{B}x - \bar{b}) + (1 - f(\bar{\mu}^T\bar{b}))(\bar{\mu}^T\bar{B}x - \lfloor \bar{\mu}^T\bar{b} \rfloor) \leq 0 \tag{7}$$

$$(\bar{\mu}^+)^T(\bar{B}x - \bar{b}) - f(\bar{\mu}^T\bar{b})(\bar{\mu}^T\bar{B}x - \lfloor \bar{\mu}^T\bar{b} \rfloor) + f(\bar{\mu}^T\bar{b}) \leq 0 \tag{8}$$

$$|\bar{\mu}|^T(\bar{B}x - \bar{b}) + (1 - 2f(\bar{\mu}^T\bar{b}))(\bar{\mu}^T\bar{B}x - \lfloor \bar{\mu}^T\bar{b} \rfloor) + f(\bar{\mu}^T\bar{b}) \leq 0^* \tag{9}$$

($y^- = \max\{-y, 0\}$, $y^+ = \max\{y, 0\}$, $f(y) = y - \lfloor y \rfloor$ and operations over vectors are componentwise).

Proof. Apply Proposition 1 to "$P = P(B)$". □

Just a convenient re-write of known properties of intersection cuts [1,2,3].

Integer Lattices and Cuts from a Mixed Integer Farkas Lemma

Definition 1. *Let* $\{v^i\}_{i\in\mathcal{V}} \subseteq \mathbb{R}^r$ *be a finite set of linear independent vectors. The* lattice *generated by* $\{v^i\}_{i\in\mathcal{V}}$ *is*

$$\mathcal{L} := \{\mu \in \mathbb{R}^r \,:\, \mu = \sum_{i\in\mathcal{V}} k_i v^i \quad k_i \in \mathbb{Z}\} \tag{10}$$

Let $\bar{B}_I \in \mathbb{R}^{r\times|N_I|}$ and $\bar{B}_C \in \mathbb{R}^{r\times(n-|N_I|)}$ be the sub-matrices of $\bar{B}$ corresponding to the integer and the continuous variables of P_I, then

Proposition 3. *[8]For every* $B \in \mathcal{B}_r^*$

$$\mathcal{L}(B) := \{\bar{\mu} \in \mathbb{R}^r \,:\, \bar{B}_I{}^T\bar{\mu} \in \mathbb{Z}^{|N_I|}, \quad \bar{B}_C{}^T\bar{\mu} = 0\} \tag{11}$$

is a lattice. If $\bar{\mu} \in \mathcal{L}(B)$ *is such that* $\bar{\mu}^T b \notin \mathbb{Z}$ *then the inequality defined by*

$$\lceil\bar{\mu}^-\rceil^T(\bar{B}x - \bar{b}) + (1 - f(\bar{\mu}^T\bar{b}))(\bar{\mu}^T\bar{B}x - \lfloor\bar{\mu}^T\bar{b}\rfloor) \leq 0 \tag{12}$$

is valid for $\{x \in P(B) \,:\, x_j \in \mathbb{Z}\,\forall\, j \in N_I\}$*. Furthermore this inequality is not satisfied by* $x(B)$*.*

Integer Lattices, Cuts from a Mixed Integer Farkas Lemma and Split Cuts

Every $\bar{\mu} \in \mathcal{L}(B)$ such that $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$ induces a split disjunction. [4]

More precisely
Proposition 4.

$$SC(B) = \bigcap_{\substack{\bar{\mu} \in \mathcal{L}(B) \\ \bar{\mu}^T \bar{b} \notin \mathbb{Z}}} \{x \in P(B) : \delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)\}.$$

Proof. Direct from Proposition 2 and definition of $SC(B)$. □

and
Proposition 5. *Let $\bar{\mu} \in \mathcal{L}(B)$ be such that $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$ then cut (12) for $\bar{\mu}$ is dominated by split cut $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$.*

Proof. From (7), $\bar{B}x - \bar{b} \leq 0$ for all $x \in P(B)$ and $\lceil \bar{\mu}^- \rceil \geq \bar{\mu}^-$. □

Polyhedrality of $SC(B)$: Preliminaries

For any $\sigma \in \{0,1\}^r$ let

$$\mathcal{L}(B,\sigma) := \{\mu \in \mathcal{L}(B) : (-1)^{\sigma_i}\mu_i \geq 0, \quad \forall\, i \in \{1,\ldots,r\}\}$$

be the intersection of $\mathcal{L}(B)$ with the orthant defined by σ, so that

$$\mathcal{L}(B) = \bigcup_{\sigma\in\{0,1\}^r} \mathcal{L}(B,\sigma)$$

Lemma 1. *Let $\sigma \in \{0,1\}^r$ and let $\bar{\mu} \in \mathcal{L}(B,\sigma)$ with $\bar{\mu} = \alpha + \beta$ for $\alpha, \beta \in \mathcal{L}(B,\sigma)$ such that $\beta^T\overline{b} \in \mathbb{Z}$. Then $\delta(\bar{\mu},B)^T x \leq \delta_0(\bar{\mu},B)$ is dominated by $\delta(\alpha,B)^T x \leq \delta_0(\alpha,B)$.*

Proof. Noting that $\lfloor\bar{\mu}^T\overline{b}\rfloor = \lfloor\alpha^T\overline{b}\rfloor + \beta^T\overline{b}$, $f(\bar{\mu}^T\overline{b}) = f(\alpha^T\overline{b})$, $|\alpha+\beta| = |\alpha| + |\beta|$ for α, β in the same orthant and using representation (9) we have that

$$\delta(\bar{\mu},B)^T x - \delta_0(\bar{\mu},B) = \delta(\alpha,B)^T x - \delta_0(\alpha,B) + f(\alpha^T\overline{b})\beta^{-T}(\bar{B}x - \overline{b}) + (1 - f(\alpha^T\overline{b}))\beta^{+T}(\bar{B}x - \overline{b}).$$

□

Polyhedrality of $SC(B)$: Preliminaries

Let $\{v^i\}_{i\in\mathcal{V}(\sigma)} \subseteq \mathcal{L}(B,\sigma)$ be a finite integral generating set of $\mathcal{L}(B,\sigma)$. That is, a finite set $\{v^i\}_{i\in\mathcal{V}(\sigma)}$ such that

$$\mathcal{L}(B,\sigma) = \{\mu \in \mathbb{R}^r : \mu = \sum_{i\in\mathcal{V}(\sigma)} k_i v^i \quad k_i \in \mathbb{Z}_+\}$$

For $i \in \mathcal{V}(\sigma)$ let

$$m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} : m\,\bar{b}^T v^i \in \mathbb{Z}\}$$

For every $\sigma \in \{0,1\}^r$ define the following finite subset of $\mathcal{L}(B,\sigma)$.

$$\mathcal{L}^0(B,\sigma) := \{\mu \in \mathcal{L}(B,\sigma) : \mu = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i \quad r_i \in \{0,\ldots,m_i-1\}\}$$

Also define the following finite subset of $\mathcal{L}(B)$.

$$\mathcal{L}^0(B) := \bigcup_{\sigma\in\{0,1\}^r} \mathcal{L}^0(B,\sigma)$$

Polyhedrality of $SC(B)$

Theorem 4. *For any $B \in B_r^*$ we have that $SC(B)$ is a polyhedron defined by the original inequalities of $P(B)$ and the following finite set of inequalities*

$$\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B) \quad \forall\, \bar{\mu} \in \mathcal{L}^0(B) \text{ s.t. } \bar{\mu}^T b \notin \mathbb{Z}.$$

Proof. For $\bar{\mu} \in \mathcal{L}(B)$, let $\sigma \in \{0,1\}^r$ be such that $\bar{\mu} \in \mathcal{L}(B,\sigma)$ and $\{k_i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+$ be such that $\bar{\mu} = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i$. For all $i \in \mathcal{V}(\sigma)$ $k_i = n_i m_i + r_i$ for some $n_i, r_i \in \mathbb{Z}_+$, $0 \leq r_i < m_i$. Let

$$\alpha = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i \quad \text{and} \quad \beta = \sum_{i \in \mathcal{V}(\sigma)} n_i m_i v^i$$

We have $\bar{\mu} = \alpha + \beta$, $\bar{b}^T \beta$ and $\bar{\mu}, \alpha, \beta \in \mathcal{L}(B,\sigma)$ so by Lemma 1 $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is dominated by $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$. The result follows from $\alpha \in \mathcal{L}^0(B,\sigma) \subseteq \mathcal{L}^0(B)$ and Proposition 4 $\square$

Corollary 2. *SC is a polyhedron.*

Final Remarks

Set of inequalities in Theorem 4 is not minimal for the description of SC or $SC(B)$. We can further require r_i's to be relatively prime.

Another constructive proof of the polyhedrality of SC based on MIR inequalities is presented in [7].

References

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