Planning Activities with Start-Time Dependent Variable Costs

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Multiple Period Planning Model

Time Horizon:	T periods
Activities to Plan:	$a \in A$
Decision Variables:	$x_a = (x_{a1}, \ldots, x_{aT}), \quad a \in A$
	$u \in \mathbb{R}^K$
Feasible Region:	$x_a \in X_a \subseteq [0, M_a]^T$
	$(x, u) \in C \subseteq \mathbb{R}^{T\|A\|+K}$

The Model

$$\min \sum_{a \in A} h_a(x_a) + \sum_k d_k u_k \tag{1}$$

$$\text{s.t.} \quad x_a \in X_a \ \ \forall a \in A \tag{2}$$

$$(u, x) \in C. \tag{3}$$

The Cost Function for Each Activity is Non-convex

c_{at} : Variable cost of activity a over the entire horizon if the activity *begins* in period t

Assumption (improving technology): $c_{a1} \geq c_{a2} \geq \cdots \geq c_{aT}$

The Cost Function

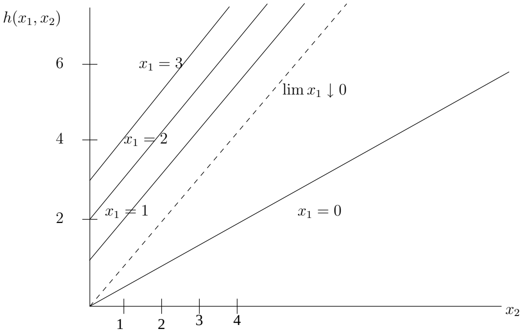
$$h_a(x) = \sum_{s=1}^{T} \mathbf{1}\left(\min\{k : x_k > 0\} = s\right) c_{as} \sum_{t=s}^{T} x_t$$

Note: h_a is concave over $\mathbb{R}_+^T$ and discontinuous.

Our Approach: Develop strong formulations for single activity; that is, for fixed a, consider formulation with cost function $h_a(x)$ and $x \in X_a$.

Example Cost Function $h(x)$ for $T = 2$

$$c_1 = 1, c_2 = \frac{1}{2},\ h(x_1, x_2) = \mathbf{1}\,(x_1 > 0)\,(x_1 + x_2) + \mathbf{1}\,(x_1 = 0)\,\frac{1}{2}x_2$$



Compact Formulation and Specialized Branching

- Introduce no auxiliary modeling variables.
- Linear lower bound on objective function:

$$h(x) \geq \sum_{t=1}^{T} c_t x_t$$

- Branching on start-time s:
  - Left branch: $s \leq k$. Update objective lower bound.
  - Right branch: $s > k$. Fix $x_t = 0$, $t = 1, \ldots, k$.
- Requires implementation of branching.
- Cuts can be used to get stronger objective lower bound.

Formulation Inspired by Lot Sizing

Introduce auxiliary variables

z_t : Amount of activity that is charged at rate c_t; i.e. amount "produced" in period t that can be used in periods $s \geq t$

All activity must be charged at rate in period activity starts

$\Rightarrow z_t$ positive in at most one period ($SOS1$)

Replace $h(x)$ with $\sum_{t=1}^{T} c_t z_t$ in objective, and add constraints:

$$\sum_{s=1}^{t} z_s \geq \sum_{s=1}^{t} x_s \qquad t = 1, \ldots, T \tag{4}$$

$$\{z_t : 1 \leq t \leq T\} \; SOS1 \tag{5}$$

Note: (5) can also be enforced by adding binary variables.

Extended Formulation Inspired by Lot Sizing

Introduce auxiliary variables

w_{st} : Amount of activity that is charged at rate c_s and performed in period $t \geq s$

Replace $h(x)$ with $\sum_{s=1}^{T} c_s \sum_{t=s}^{T} w_{st}$ in objective, and add:

$$\sum_{s=1}^{t} w_{st} = x_t \qquad t = 1, \ldots, T \tag{6}$$

$$w_{st} \leq M b_s \qquad s = 1, \ldots, T \tag{7}$$

$$\sum_{t=1}^{T} b_t \leq 1 \tag{8}$$

$$b_t \in \{0, 1\} \qquad t = 1, \ldots, T \tag{9}$$

Note: For a single activity, this formulation is integral.

Strengthening the Formulations: A Special Case

- $X = \{x : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_T \leq M\}$.
- The nondecreasing constraint was present in the motivating application.
- Compact formulation: Improved lower bound

$$h(x) \geq \sum_{t=1}^{T} c_t x_t + \sum_{t=1}^{T-1} (T-t)(c_t - c_{t+1}) x_t \qquad (10)$$

- Lot sizing inspired formulation: Strengthen inequalities (4)

$$\sum_{s=1}^{t} z_s \geq \sum_{s=1}^{t} x_s + (T-t) x_t \qquad t = 1, \ldots, T \qquad (11)$$

Strengthening the Formulations: A Special Case

Theorem

$$conv(F) = E = Proj_{(\mu,x)}(P)$$

where $F = \{(\mu, x) \in \mathbb{R} \times X : \mu \geq h(x)\}$ is the non-convex feasible region (the epigraph of non-convex function h),

$$E = \{(\mu, x) \in \mathbb{R} \times X : \mu \geq \sum_{t=1}^{T} c_t x_t + \sum_{t=1}^{T-1} (T - t)(c_t - c_{t+1}) x_t\}$$

is the strengthened compact formulation, and

$$P = \{(\mu, x, z) \in \mathbb{R} \times X \times \mathbb{R}_+^T : \mu = \sum_{t=1}^{T} c_t z_t, (x, z) \text{ satsify } (11)\}$$

is the strengthened lot sizing inspired formulation.

Select Computational Results: Production and Distribution Planning

- Minimize costs to meet demand over the planning horizon. Production and distribution costs are start-time dependent.
- Instances randomly generated, with characteristics similar to real data.

Results for lot sizing formulation with binary variables: strengthening the formulation with (11) is crucial.

		Time(s) or * Gap		Nodes	
$\|A\|$	T	Ineqs (4)	Ineqs (11)	Ineqs (4)	Ineqs (11)
100	10	50.3	8.2	451	5
150	10	* 0.08%	13.4	75488	76
200	10	* 0.09%	45.2	35951	242
75	15	992.6	9.9	29738	39

* Did not finish after limit of 1 hour.

Good Solutions Can be Found for Large Instances

		Time(s) or * Gap			Nodes		
$\|A\|$	T	No-A	LS-S	LS-B	No-A	LS-S	LS-B
300	10	129.4	1786.8	293.0	4410	8683	1651
400	10	24.8	209.3	106.2	454	873	268
500	10	27.7	748.5	270.3	1296	2567	779
200	15	699.9	* 0.05%	* 0.10%	6775	3432	1302
300	15	1335.5	* 0.07%	* 0.02%	12606	1545	8750
400	15	234.5	* 0.06%	2294.6	5612	736	3828
500	20	* 0.02%	* 0.07%	* 0.06%	40128	482	903
1000	20	* 0.08%	* 0.38%	* 0.18%	6891	20	0

* Did not finish after limit of 1 hour.

No-A = Compact formulation, LS-S = Lot sizing with SOS1,
LS-B = Lot sizing with binaries
Compact formulation has significantly faster LP solve times.

The Approach Can Also Handle Side Constraints

- Add semi-continuous restrictions on activities: $x_t \in \{0\} \cup [l, M]$
- For compact and lot sizing with SOS1, binaries added only to model this restriction

		Time(s) or * Gap			Nodes		
$\|A\|$	T	No-Aux	LS-S	LS-B	No-Aux	LS-S	LS-B
100	10	4.6	20.0	46.2	77	18	165
200	10	455.7	397.9	1585.0	2822	1058	5129
300	10	* 0.79%	* 0.48%	* 0.57%	5760	1868	5229
400	10	* 0.26%	* 0.20%	* 0.18%	7828	1840	5196
200	15	* 0.97%	* 1.01%	* 0.95%	2640	725	960
300	15	* 0.49%	* 0.56%	* 0.44%	5078	1601	899
400	15	* 0.20%	* 0.29%	* 0.12%	6331	733	2051

For larger instances, formulation with binaries yields better gap within time limit.

Extensions and Ongoing Research

- Remove the non-decreasing constraint on activities. That is, let
$$X = \left\{ x \in \mathbb{R}_+^T : x_t \leq M,\ t = 1, \ldots, T \right\}.$$
Single activity convex hull defined by exponential family of inequalities in all formulations (except extended).
- Incorporate fixed cost for installing technology. Motivates further study of models with binary variables present.
- Consider more complicated sets X. For example, time-dependent upper bounds, or production ramping constraints.